ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}\ge 5$$

*Proposed by Adil Abdullayev-Azerbaijan*

*Solution 1 by Soumava Chakraborty-Kolkata-India*

$$\text{Tereshin} \Rightarrow m_a \ge \frac{b^2+c^2}{4R} \Rightarrow \frac{4RFm_a}{F} \ge b^2+c^2 \Rightarrow \frac{abcm_a}{F} \ge b^2+c^2$$

$\Rightarrow \frac{am_a}{F} \ge \frac{b}{c}+\frac{c}{b}$ implementing which on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose area via elementary calculations $=\frac{F}{3}$ and medians $=\frac{a}{2},\frac{b}{2},\frac{c}{2}$,

$$\text{we get : } \frac{\left(\frac{2m_a}{3}\right)\left(\frac{a}{2}\right)}{\frac{F}{3}} \ge \frac{\frac{2m_b}{3}}{\frac{2m_c}{3}}+\frac{\frac{2m_c}{3}}{\frac{2m_b}{3}} \Rightarrow \frac{2m_a}{\left(\frac{2F}{a}\right)} \ge \frac{m_b}{m_c}+\frac{m_c}{m_b} \Rightarrow \frac{2m_a}{h_a} \ge \frac{m_b}{m_c}+\frac{m_c}{m_b} \to (1)$$

$$\therefore \text{ via (1) and analogs, } \sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}} \ge \sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

$$=\sum_{cyc}\frac{m_b^2+m_c^2}{\sqrt{2m_bm_c(m_b^2+m_c^2)}} \ge \sum_{cyc}\frac{2(m_b^2+m_c^2)}{\sqrt{8m_bm_c(m_b^2+m_c^2)}} = \sum_{cyc}\frac{2(m_b^2+m_c^2)}{(m_b+m_c)^2}$$

$$\left(\because (y+z)^4=\left(y^2+z^2+2yz\right)^2 \overset{A-G}{\ge} 8yz(y^2+z^2)\right)$$

$$\Rightarrow \sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}$$

$$\ge \sum_{cyc}\frac{2(m_b^2+m_c^2)}{(m_b+m_c)^2}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2} \overset{?}{\ge} 5$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{2(y^2+z^2)}{(y+z)^2}-1\right) \overset{?}{\ge} 2-\frac{6\sum_{cyc}xy}{\left(\sum_{cyc}x\right)^2} \quad (x=m_a, y=m_b, z=m_c)$$

$$\Leftrightarrow \boxed{\sum_{cyc}\frac{(y-z)^2}{(y+z)^2} \underset{(*)}{\overset{?}{\ge}} \frac{2\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)}{\left(\sum_{cyc}x\right)^2}}$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X,Y,Z$ form sides of triangle with semiperimeter, circumradius and inradius $=s',R',r'$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s' \Rightarrow \sum_{cyc}x=s' \to (1) \Rightarrow x=s'-X, y=s'-Y,$ $z=s'-Z$ and suchsubstitutions $\Rightarrow xyz=r'^2s' \to (2)$,

$$\sum_{cyc} xy = \sum_{cyc}(s'-X)(s'-Y) \Rightarrow \sum_{cyc} xy = 4R'r' + r'^2 \to (3) \text{ and}$$

$$\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (3)}}{=} s'^2 - 2(4R'r' + r'^2)$$

$$\Rightarrow \sum_{cyc} x^2 = s'^2 - 8R'r' - 2r'^2 \to (4) \therefore (*)$$

$$\Leftrightarrow \sum_{cyc} \frac{\left((s'-Y)-(s'-Z)\right)^2}{X^2} \geq \frac{2\left(s'^2 - 8R'r' - 2r'^2 - (4R'r' + r'^2)\right)}{s'^2}$$

$$\Leftrightarrow \boxed{\sum_{cyc} \frac{(Y-Z)^2}{X^2} \overset{(**)}{\geq} \frac{2(s'^2 - 12R'r' - 3r'^2)}{s'^2}}$$

$$\text{Now, } \sum_{cyc} b^2c^2(b-c)^2 = \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 3a^2b^2c^2 - 2\sum_{cyc} a^3b^3$$

$$= \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 3a^2b^2c^2$$

$$-2\left(3a^2b^2c^2 + \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right)\right)$$

$$= 2\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right)\left((s^2-4Rr-r^2) - (s^2+4Rr+r^2)\right) - 144R^2r^2s^2$$

$$+2.4Rrs.2s(s^2+4Rr+r^2) = 4r^2\left((12R^2+4Rr-2r^2)s^2 - s^4 - r(4R+r)^3\right)$$

$$\Rightarrow \sum_{cyc} \frac{(b-c)^2}{a^2} = \frac{\sum_{cyc} b^2c^2(b-c)^2}{16R^2r^2s^2}$$

$$= \frac{4r^2\left((12R^2+4Rr-2r^2)s^2 - s^4 - r(4R+r)^3\right)}{16R^2r^2s^2} \overset{?}{\geq} \frac{2(s^2-12Rr-3r^2)}{s^2}$$

$$\Leftrightarrow \boxed{s^4 - (4R^2+4Rr-2r^2)s^2 - r(32R^3-24R^2r-12Rr^2-r^3) \underset{(***)}{\overset{?}{\leq}} 0}$$

$$\text{Now, LHS of } (***) \overset{\text{Gerretsen}}{\leq} \left((4R^2+4Rr+3r^2) - (4R^2+4Rr-2r^2)\right)s^2$$

$$-r(32R^3-24R^2r-12Rr^2-r^3) = 5r^2s^2 - r(32R^3-24R^2r-12Rr^2-r^3)$$

$$\overset{\text{Gerretsen}}{\leq} 5r^2(4R^2+4Rr+3r^2) - r(32R^3-24R^2r-12Rr^2-r^3) \overset{?}{\leq} 0$$

$$\Leftrightarrow 8t^3 - 11t^2 - 8t - 4 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(8t^2+5t+2) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true} \therefore \sum_{cyc} \frac{(b-c)^2}{a^2} \geq \frac{2(s^2-12Rr-3r^2)}{s^2}$$

$$\Rightarrow \sum_{cyc} \frac{(Y-Z)^2}{X^2} \overset{(**)}{\geq} \frac{2\left(s'^2 - 12R'r' - 3r'^2\right)}{s'^2} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} + \frac{6(m_a m_b + m_b m_c + m_c m_a)}{(m_a + m_b + m_c)^2} \geq 5$$

$\forall \Delta\, ABC,'' ='' \text{ iff } \Delta\, ABC$ is equilateral (QED)

## *Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco*

**By Tereshin's inequality, we have**

$$\frac{m_a}{h_a} \geq \frac{b^2 + c^2}{4Rh_a} = \frac{b^2 + c^2}{2bc}.$$

**Applying this inequality to the triangle $GBC$**
**($G$ is the centroid of the triangle $ABC$) and noting that**
**the altitude from $G$ is equal to $\frac{h_a}{3}$ and $GB = \frac{2}{3}m_b$, $GC = \frac{2}{3}m_c$, we get**

$$\frac{\frac{m_a}{3}}{\frac{h_a}{3}} \geq \frac{\left(\frac{2}{3}m_b\right)^2 + \left(\frac{2}{3}m_c\right)^2}{2.\frac{2}{3}m_b.\frac{2}{3}m_c} \quad \text{or} \quad \frac{m_a}{h_a} \geq \frac{m_b^2 + m_c^2}{2m_b m_c}.$$

$$\Rightarrow \sqrt{\frac{m_a}{h_a}} \geq \sqrt{\frac{m_b^2 + m_c^2}{2m_b m_c}} = \frac{2(m_b^2 + m_c^2)}{2\sqrt{2m_b m_c(m_b^2 + m_c^2)}} \overset{AM-GM}{\geq} \frac{2(m_b^2 + m_c^2)}{2m_b m_c + (m_b^2 + m_c^2)}$$

$$= \frac{(m_b - m_c)^2}{(m_b + m_c)^2} + 1 \geq \frac{(m_b - m_c)^2}{(m_a + m_b + m_c)^2} + 1 \ \ \text{(and analogs)}$$

**Therefore**

$$\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} + \frac{6(m_a m_b + m_b m_c + m_c m_a)}{(m_a + m_b + m_c)^2} \geq$$

$$\geq \frac{(m_b - m_c)^2 + (m_c - m_a)^2 + (m_a - m_b)^2 + 6(m_a m_b + m_b m_c + m_c m_a)}{(m_a + m_b + m_c)^2} + 3 = 5,$$

**as desired. Equality holds iff $\Delta ABC$ is equilateral.**