ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a m_b m_c}{r_a r_b r_c} \le \frac{R}{2r}$$

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$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32\left(s^2-4Rr-r^2\right)^3+36\left(s^2-4Rr-r^2\right)\left(s^2+4Rr+r^2\right)^2\\-576Rrs^2\left(s^2-4Rr-r^2\right)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\left\{s^6-s^4\left(12Rr-33r^2\right)-s^2\left(60R^2r^2+120Rr^3+33r^4\right)-r^3(4R+r)^3\right\}$$

$$\leq\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4\left(4R^2+12Rr-33r^2\right)-s^2\left(60R^2r^2+120Rr^3+33r^4\right)-r^3(4R+r)^3\overset{(\bullet)}{\leq}0$$

$$\text{Now, LHS of }(\bullet)\overset{\text{Gerretsen}}{\leq}-s^4\left(8Rr-36r^2\right)-s^2\left(60R^2r^2+120Rr^3+33r^4\right)$$

$$-r^3(4R+r)^3\overset{?}{\leq}0$$

$$\Leftrightarrow s^4(8R-16r)+s^2\left(60R^2r+120Rr^2+33r^3\right)+r^2(4R+r)^3\underbrace{\overset{?}{\geq}}_{(\bullet\bullet)}20rs^4$$

$$\text{Now, LHS of }(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)}s^2\left(16Rr-5r^2\right)(8R-16r)$$

$$+s^2\left(60R^2r+120Rr^2+33r^3\right)+r^2(4R+r)^3\text{ and}$$

$$\text{RHS of }(\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)}20rs^2\left(4R^2+4Rr+3r^2\right)$$

$(a),(b)\Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2\left(16Rr-5r^2\right)(8R-16r)+s^2\left(60R^2r+120Rr^2+33r^3\right)+r^2(4R+r)^3$$

$$\geq 20rs^2\left(4R^2+4Rr+3r^2\right)$$

$$\Leftrightarrow s^2\left(108R^2-256Rr+53r^2\right)+r(4R+r)^3\geq 0$$

$$\Leftrightarrow s^2\left(108R^2-256Rr+80r^2\right)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\geq}27r^2s^2$$

$$\text{Now, LHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)}\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3$$

$$\text{and RHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)}27r^2\left(4R^2+4Rr+3r^2\right)$$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$\left(108R^2-256Rr+80r^2\right)\left(16Rr-5r^2\right)+r(4R+r)^3\geq 27r^2\left(4R^2+4Rr+3r^2\right)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\ \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big)\geq 0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$

$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\leq\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\leq\frac{Rs^2}{2}\Rightarrow\frac{m_am_bm_c}{r_ar_br_c}\leq\frac{\frac{Rs^2}{2}}{rs^2}$$

$$\Rightarrow\frac{m_am_bm_c}{r_ar_br_c}\leq\frac{R}{2r}\ \forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$