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In ΔABC the following relationships holds :

$$\sqrt[5]{\frac{s_a}{m_a}}+\sqrt[5]{\frac{s_b}{m_b}}+\sqrt[5]{\frac{s_c}{m_c}}+\frac{r_ar_br_c}{w_aw_bw_c}\ge 4$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $\frac{w_a}{\sqrt{r_br_c}}=\frac{2\sqrt{bc}}{b+c}\le 1$ (and analogs).

By using the AM − GM Inequality, we have

$$\frac{s_a}{m_a}=\frac{2bc}{b^2+c^2}=\frac{16b^2c^2}{4.2bc.(b^2+c^2)}\ge\frac{16b^2c^2}{[2bc+(b^2+c^2)]^2}=\left(\frac{2\sqrt{bc}}{b+c}\right)^4\ge$$

$$\ge\left(\frac{2\sqrt{bc}}{b+c}\right)^5=\left(\frac{w_a}{\sqrt{r_br_c}}\right)^5\Rightarrow\sqrt[5]{\frac{s_a}{m_a}}\ge\frac{w_a}{\sqrt{r_br_c}}\quad\text{(and analogs)}$$

Therefore

$$\sqrt[5]{\frac{s_a}{m_a}}+\sqrt[5]{\frac{s_b}{m_b}}+\sqrt[5]{\frac{s_c}{m_c}}+\frac{r_ar_br_c}{w_aw_bw_c}\ge\frac{w_a}{\sqrt{r_br_c}}+\frac{w_b}{\sqrt{r_cr_a}}+\frac{w_c}{\sqrt{r_ar_b}}+\frac{r_ar_br_c}{w_aw_bw_c}$$

$$\overset{AM-GM}{\ge} 4\sqrt[4]{\frac{w_a}{\sqrt{r_br_c}}.\frac{w_b}{\sqrt{r_cr_a}}.\frac{w_c}{\sqrt{r_ar_b}}.\frac{r_ar_br_c}{w_aw_bw_c}}=4.$$

Equality holds iff ΔABC is equilateral.