ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{4a^2+b^2+c^2}{2a^2+bc}\le\frac{R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$R-2r\overset{?}{\ge}\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\overset{?}{\ge}\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}$$

$$\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\overset{?}{\ge}\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\overset{?}{\ge}\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\overset{?}{\ge}4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\overset{?}{\ge}4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\overset{?}{\ge}0\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\overset{?}{\ge}0$$

$$\to \text{true}\Rightarrow(b-c)^2\le 4R(R-2r)\therefore\frac{4a^2+b^2+c^2-\left(4a^2+2bc\right)}{4a^2+2bc}\le\frac{4R(R-2r)}{4a^2+2bc}$$

$$\overset{?}{\le}\frac{R}{2r}-1=\frac{R-2r}{2r}\Leftrightarrow 8Rrs\overset{?}{\le}s\left(4a^2+2bc\right)\ \left(\because R-2r\overset{\text{Euler}}{\ge}0\right)$$

$$\Leftrightarrow 2abc\overset{?}{\le}(a+b+c)\left(2a^2+bc\right)$$

$$\Leftrightarrow 2a^3+abc+2a^2b+b^2c+2a^2c+bc^2\overset{?}{\ge}2abc$$

$$\Leftrightarrow 2a^3+2a^2b+b^2c+2a^2c+bc^2\overset{?}{\ge}abc\to\text{true}\because a^3+b^2c+bc^2\overset{A-G}{\ge}3abc$$

$$>abc\Rightarrow 2a^3+2a^2b+b^2c+2a^2c+bc^2>abc$$

$$\therefore\frac{4a^2+b^2+c^2-\left(4a^2+2bc\right)}{4a^2+2bc}\le\frac{R}{2r}-1\Rightarrow\frac{4a^2+b^2+c^2}{4a^2+2bc}-1\le\frac{R}{2r}-1$$

$$\Rightarrow\frac{4a^2+b^2+c^2}{2a^2+bc}\le\frac{R}{r}\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$