ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{12\sqrt{3}F}{a^2+b^2+c^2}\le 4$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{We shall prove : }\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{9\sqrt{3}F}{m_a^2+m_b^2+m_c^2}\le 4\to(1)$$

$$\text{LHS of }(1)\overset{\text{Mitrinovic}}{\le}\frac{2(s^2-4Rr-r^2)}{s^2+4Rr+r^2}+\frac{3s^2}{\frac{3}{2}(s^2-4Rr-r^2)}\overset{?}{\le}4$$

$$\Leftrightarrow 2-\frac{s^2-4Rr-r^2}{s^2+4Rr+r^2}\overset{?}{\ge}\frac{s^2}{s^2-4Rr-r^2}\Leftrightarrow\frac{s^2+12Rr+3r^2}{s^2+4Rr+r^2}\overset{?}{\ge}\frac{s^2}{s^2-4Rr-r^2}$$

$$\Leftrightarrow(4R+r)s^2\overset{?}{\ge}3r(4R+r)^2\Leftrightarrow s^2-12Rr-3r^2\overset{?}{\ge}0$$

$$\Leftrightarrow s^2-16Rr+5r^2+4r(R-2r)\overset{?}{\ge}0\to\text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0$$

$$\text{and }R-2r\overset{\text{Euler}}{\ge}0\therefore\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{9\sqrt{3}F}{m_a^2+m_b^2+m_c^2}\le 4\text{ and implementing }(1)$$

$$\text{on a triangle with sides }\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}\text{ whose area}$$

$$\text{via elementary calculations }=\frac{F}{3}\text{ and medians }=\frac{a}{2},\frac{b}{2},\frac{c}{2}\text{ we get :}$$

$$\frac{\frac{4}{9}(m_a^2+m_b^2+m_c^2)}{\frac{4}{9}(m_am_b+m_bm_c+m_cm_a)}+\frac{\frac{9\sqrt{3}F}{3}}{\frac{1}{4}(a^2+b^2+c^2)}\le 4$$

$$\Rightarrow\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{12\sqrt{3}F}{a^2+b^2+c^2}\le 4$$

$$\forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$m_am_b+m_bm_c+m_cm_a\overset{AM-GM}{\ge}3\sqrt[3]{(m_am_bm_c)^2}\overset{Leuenberger}{\ge}3\sqrt[3]{(s^2r)^2}\overset{Mitrinovic}{\ge}3\sqrt{3}F,$$

and since $a^2+b^2+c^2=\frac{4}{3}(m_a{}^2+m_b{}^2+m_c{}^2)$, then it suffices to prove that

$$\frac{m_a{}^2+m_b{}^2+m_c{}^2}{m_am_b+m_bm_c+m_cm_a}+\frac{3(m_am_b+m_bm_c+m_cm_a)}{m_a{}^2+m_b{}^2+m_c{}^2}\le 4 \qquad (1)$$

$$\text{Let } x:=\frac{m_a{}^2+m_b{}^2+m_c{}^2}{m_am_b+m_bm_c+m_cm_a}.$$

Since m_a, m_b, m_c can be the sides of a triangle, we have $1\le x<2$.

$$\Rightarrow\ (1)\ \Leftrightarrow x+\frac{3}{x}\le 4\ \Leftrightarrow\ \frac{(x-1)(3-x)}{x}\ge 0,$$

which is true. So the proof is complete. Equality holds iff ΔABC is equilateral.