ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{6(a^2b^2+b^2c^2+c^2a^2)}{(a^2+b^2+c^2)^2} \geq 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{6(a^2b^2+b^2c^2+c^2a^2)}{(a^2+b^2+c^2)^2}-1$$
$$=\frac{3\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-2\left(s^2-4Rr-r^2\right)^2}{2(s^2-4Rr-r^2)^2}$$
$$=\frac{s^4-(8Rr-10r^2)s^2+r^2(4R+r)^2}{2(s^2-4Rr-r^2)^2} \geq \frac{4\sqrt{3}F}{a^2+b^2+c^2}=\frac{2\sqrt{3}rs}{s^2-4Rr-r^2}$$
$$\Leftrightarrow \frac{\left(s^4-(8Rr-10r^2)s^2+r^2(4R+r)^2\right)^2}{4(s^2-4Rr-r^2)^4} \geq \frac{12r^2s^2}{(s^2-4Rr-r^2)^2}$$
$$\Leftrightarrow \left(s^4-(8Rr-10r^2)s^2+r^2(4R+r)^2\right)^2 \overset{(*)}{\geq} 48r^2s^2\left(s^2-4Rr-r^2\right)^2 \text{ and}$$
$$\because \left(s^2-16Rr+5r^2\right)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (*), \text{it suffices to prove :}$$
$$\text{LHS of } (*) \geq \left(s^2-16Rr+5r^2\right)^4$$
$$\Leftrightarrow (R-r)s^6-r(30R^2-25Rr-r^2)s^4+r^2(336R^3-332R^2r+95Rr^2-11r^3)s^2$$
$$-r^3(1360R^4-1712R^3r+798R^2r^2-167Rr^3+13r^4) \overset{(**)}{\geq} 0$$
$$\text{and} \because (R-r)\left(s^2-16Rr+5r^2\right)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (**),$$
$$\text{it suffices to prove : LHS of } (**) \geq (R-r)\left(s^2-16Rr+5r^2\right)^3$$
$$\Leftrightarrow (9R^2-19Rr+8r^2)s^4-r(216R^3-458R^2r+230Rr^2-32r^3)s^2$$
$$+r^2(1368R^4-3112R^3r+2121R^2r^2-579Rr^3+56r^4) \overset{(***)}{\geq} 0 \text{ and}$$
$$\because (9R^2-19Rr+8r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (***),$$
$$\text{it suffices to prove : LHS of } (***) \geq (9R^2-19Rr+8r^2)\left(s^2-16Rr+5r^2\right)^2$$
$$\Leftrightarrow (3R^3-10R^2r+9Rr^2-2r^3)s^2 \geq r(39R^4-133R^3r+133R^2r^2-49Rr^3+6r^4)$$
$$\Leftrightarrow (R-2r)(3R^2-4Rr+r^2)s^2 \geq r(R-2r)(39R^3-55R^2r+23Rr^2-3r^3)$$
$$\Leftrightarrow (3R^2-4Rr+r^2)s^2 \overset{(****)}{\geq} r(39R^3-55R^2r+23Rr^2-3r^3) \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)$$
$$\text{Now, } (3R^2-4Rr+r^2)s^2 \overset{\text{Gerretsen}}{\geq} (3R^2-4Rr+r^2)(16Rr-5r^2) \overset{?}{\geq}$$
$$r(39R^3-55R^2r+23Rr^2-3r^3) \Leftrightarrow 9t^3-24t^2+13t-2 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(3t-1)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{6(a^2b^2+b^2c^2+c^2a^2)}{(a^2+b^2+c^2)^2} \geq 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}$$
$$\forall \Delta \text{ABC}, \text{''}=\text{'' iff } \Delta \text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since

$$16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4),$$

then the given inequality is equivalent to

$$\frac{3[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)]}{2(a^2 + b^2 + c^2)^2} + \frac{1}{2} \geq \frac{4\sqrt{3}F}{a^2 + b^2 + c^2}$$

$$\text{or } \frac{24F^2}{(a^2 + b^2 + c^2)^2} + \frac{1}{2} \geq \frac{4\sqrt{3}F}{a^2 + b^2 + c^2} \text{ or } \frac{1}{2}\left(\frac{4\sqrt{3}F}{a^2 + b^2 + c^2} - 1\right)^2 \geq 0,$$

which is true. Equality holds iff $a^2 + b^2 + c^2 = 4\sqrt{3}F$ or ΔABC is equilateral.