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In $\Delta ABC$ the following relationship holds:

$$\frac{18R^2}{a^2+b^2+c^2} \geq 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}$$

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*Solution by Daniel Sitaru-Romania*

$$\frac{18R^2}{a^2+b^2+c^2} \geq 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2} \Leftrightarrow a^2+b^2+c^2+4\sqrt{3}F \leq 18R^2$$

Nakajima(1925): In $\Delta ABC$ the following relationship holds:

$$a^2+b^2+c^2 \leq 8R^2+\frac{4}{3\sqrt{3}}F$$

Proof:

$$a^2+b^2+c^2 = (a+b+c)^2-2(ab+bc+ca) =$$

$$= 4s^2-2(s^2+r^2+4Rr) = 2s^2-2r^2-8Rr \overset{GERRETSEN}{\leq}$$

$$\leq 2(4R^2+4Rr+3r^2)-2r^2-8Rr = 8R^2+4r^2 = 8R^2+4r\cdot r \leq$$

$$\overset{MITRINOVIC}{\leq} 8R^2+4r\cdot\frac{s}{3\sqrt{3}} = 8R^2+\frac{4}{3\sqrt{3}}F$$

Back to the problem:

$$a^2+b^2+c^2+4\sqrt{3}F \overset{Nakajima}{\leq} 8R^2+\frac{4}{3\sqrt{3}}F+4\sqrt{3}F$$

Remains to prove:

$$8R^2+\frac{4}{3\sqrt{3}}F+4\sqrt{3}F \leq 18R^2 \Leftrightarrow \left(\frac{4}{3\sqrt{3}}+4\sqrt{3}\right)F \leq 10R^2$$

$$\frac{40F}{3\sqrt{3}} \leq 10R^2 \Leftrightarrow 4F \leq 3\sqrt{3}R^2 \ (to\ prove)$$

$$4F = 4rs \overset{EULER}{\leq} 2Rs \overset{MITRINOVIC}{\leq} 2R\cdot\frac{3\sqrt{3}R}{2} = 3\sqrt{3}R^2$$

Equality holds for $a=b=c$.