ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r_a^3+r_b^3+r_c^3}{r_a r_b r_c}+5\ge\frac{4R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{4R}{r}=\frac{4R+r-\frac{rs^2}{s^2}}{\frac{rs^2}{s^2}}=\frac{\sum_{cyc}r_a-\frac{r_ar_br_c}{\sum_{cyc}r_ar_b}}{\frac{r_ar_br_c}{\sum_{cyc}r_ar_b}}=\frac{\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-xyz}{xyz}$$

$$(x=r_a,y=r_b,z=r_c)\overset{?}{\le}\frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+5=\frac{\sum_{cyc}x^3+5xyz}{xyz}$$

$$\Leftrightarrow\sum_{cyc}x^3+5xyz\overset{?}{\ge}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-xyz=\sum_{cyc}x^2y+\sum_{cyc}xy^2+2xyz$$

$$\Leftrightarrow\sum_{cyc}x^3+3xyz\overset{?}{\ge}\sum_{cyc}x^2y+\sum_{cyc}xy^2\rightarrow\text{true via Schur}$$

$$\therefore\frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+5\ge\frac{4R}{r}\ \forall\Delta\text{ ABC},''=''\ \text{iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$\sum r_a^3=\left(\sum r_a\right)^3-3\left[\left(\sum r_a\right)\left(\sum r_ar_b\right)-\prod r_a\right]=(4R+r)^3-12s^2R.$$

$$Now\frac{\sum r_a^3}{r_ar_br_c}+5=\frac{(4R+r)^3-12s^2R}{s^2r}+5\overset{Gerrestn}{\ge}\frac{(4R+r)^3}{(4R^2+4Rr+3r^2)r}-\frac{12R}{r}+5,$$

we need to show

$$\frac{(4R+r)^3}{(4R^2+4Rr+3r^2)r}-\frac{12R}{r}+5\ge\frac{4R}{r}\ or\ \frac{(4R+r)^3}{(4R^2+4Rr+3r^2)r}+5\ge\frac{16R}{r}$$

$$or\frac{(4x+1)^3}{4x^2+4x+3}+5\ge16x\left(where\frac{R}{r}=x\ge2\ Euler\right)or$$

$$4x^2-16x+16\ge0\ or\ (2x-4)^2\ge0\ True$$