ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{(2R+r-r_a)(3R+r-r_b)+R^2}\ge\frac{1}{R^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum(2R+r-r_a)(3R+r-r_b)=$$

$$=\sum(6R^2+5Rr-2Rr_b-3Rr_a-r(r_a+r_b)+r_ar_b+r^2)=$$

$$=3.6R^2+3.5Rr-2R\sum r_b-3R\sum r_a-2r\sum r_a+\sum r_ar_b+3r^2=$$

$$=18R^2+15Rr-2R(4R+r)-3R(4R+r)-2r(4R+r)+s^2+3r^2=$$

$$=-2R^2+2Rr+r^2+s^2\overset{Gerretsen}{\le}-2R^2+2Rr+r^2+4R^2+4Rr+3r^2=$$

$$=2R^2+6Rr+4r^2\overset{Euler}{\le}2R^2+6R.\frac{R}{2}+4\left(\frac{R}{2}\right)^2=6R^2\ (1)$$

$$\sum_{cyc}\frac{1}{(2R+r-r_a)(3R+r-r_b)+R^2}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(1+1+1)^2}{\sum(2R+r-r_a)(3R+r-r_b)+3R^2}\overset{(1)}{\ge}\frac{9}{6R^2+3R^2}=\frac{1}{R^2}$$

$$Equality\ holds\ for\ a=b=c$$