ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{h_a\sqrt{h_a}}{w_a\sqrt{r_a}} \ge 3\sqrt{\frac{2r}{R}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$h_a h_b h_c = \frac{8F^3}{abc} = \frac{8r^3s^3}{4Rrs} = \frac{2r^2s^2}{R} \quad (1)$$

$$r_a r_b r_c = s^2 r \ (2),\ w_a w_b w_c \overset{w_a \le \sqrt{s(s-a)}}{\le} \sqrt{\prod s(s-a)} = \sqrt{s^3 s r^2} = s^2 r, \qquad (3)$$

$$\sum \frac{h_a\sqrt{h_a}}{w_a\sqrt{r_a}} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{h_a h_b h_c\sqrt{h_a h_b h_c}}{w_a w_b w_c\sqrt{r_a r_b r_c}}} \overset{(1),(2),(3)}{\ge}$$

$$\ge 3\sqrt[3]{\frac{(h_a h_b h_c)^{\frac{3}{2}}}{(s^2 r)^{\frac{3}{2}}}} = 3\sqrt{\frac{h_a h_b h_c}{s^2 r}}\, 3\sqrt{\frac{\frac{2r^2s^2}{R}}{s^2 r}} = 3\sqrt{\frac{2r}{R}}$$

Equality holds for an equilateral triangle