ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, following relationship holds :

$$\sin A \,.\, m_a w_a > F \geq 3\sqrt{3}r^2 + \sqrt{2}r(R-2r)$$

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$$F \geq 3\sqrt{3}r^2 + \sqrt{2}.r(R-2r) \Leftrightarrow s - \sqrt{2}(R-2r) \geq 3\sqrt{3}r$$

$$\Leftrightarrow \left(s - \sqrt{2}(R-2r)\right)^2 \geq 27r^2$$

$$\left(\begin{array}{c} \because \Delta \text{ ABC is acute} \Rightarrow \prod_{cyc} \cos A > 0 \Rightarrow s > 2R + r > 2R \\ \Rightarrow s - \sqrt{2}(R-2r) > \left(2-\sqrt{2}\right)R + 2\sqrt{2}.r > 0 \end{array}\right)$$

$$\Leftrightarrow s^2 + 2(R-2r)^2 - 2\sqrt{2}.s(R-2r) \geq 27r^2$$

$$\Leftrightarrow s^2 + 2(R-2r)^2 - 27r^2 \geq 2\sqrt{2}.s(R-2r) \Leftrightarrow \left(s^2 + 2(R-2r)^2 - 27r^2\right)^2$$

$$\geq 8s^2(R-2r)^2 \left(\because s^2 - 27r^2 + 2(R-2r)^2 \overset{\text{Mitrinovic}}{\geq} 2(R-2r)^2 \geq 0\right)$$

$$\Leftrightarrow s^4 - (4R^2 - 16Rr + 70r^2)s^2 + 4R^4 - 32R^3r - 12R^2r^2 + 304Rr^3 + 361r^4 \boxed{\overset{(*)}{\geq}} 0$$

and $\because (s^2 - 2R^2 - 8Rr - 3r^2)^2 \geq 0 \therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*) \geq (s^2 - 2R^2 - 8Rr - 3r^2)^2$

$$\Leftrightarrow (4R-8r)s^2 \boxed{\overset{(**)}{\geq}} 8R^3 + 11R^2r - 32Rr^2 - 44r^3$$

Now, $\because \Delta$ ABC is acute $\therefore (4R-8r)s^2 \overset{Walker}{\geq} (4R-8r)(2R^2 + 8Rr + 3r^2) \overset{?}{\geq}$
$8R^3 + 11R^2r - 32Rr^2 - 44r^3 \Leftrightarrow 40r^2(R^2 - 4Rr + 4r^2) \overset{?}{\geq} 0 \Leftrightarrow 40r^2(R-2r)^2 \overset{?}{\geq} 0$
$\rightarrow$ true $\Rightarrow (**) \Rightarrow (*)$ is true $\therefore F \geq 3\sqrt{3}r^2 + \sqrt{2}.r(R-2r)\ \forall$ acute Δ ABC

Again, $\sin A \,.\, m_a w_a - F \overset{\text{Lascu + A-G}}{\geq} \dfrac{a}{2R}.s(s-a) - \dfrac{abc}{4R} = \dfrac{a}{2R}.(2s(s-a) - bc)$

$= \dfrac{abc}{2R}.\left(2\cos^2\dfrac{A}{2} - 1\right) > 0 \because \Delta$ ABC is acute $\Rightarrow 0 < \dfrac{A}{2} < \dfrac{\pi}{4} \Rightarrow \cos\dfrac{A}{2} > \dfrac{1}{\sqrt{2}}$

$\Rightarrow 2\cos^2\dfrac{A}{2} - 1 > 0 \therefore \sin A \,.\, m_a w_a > F\ \forall$ acute Δ ABC and so,

$$\sin A \,.\, m_a w_a > F \geq 3\sqrt{3}r^2 + \sqrt{2}.r(R-2r)\ \forall \text{ acute } \Delta \text{ ABC}$$

$'' = ''$ iff Δ ABC is equilateral (QED)