ROMANIAN MATHEMATICAL MAGAZINE

In acute triangle ABC :

$$r_a < \frac{5}{2}R$$

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$$r_a < \frac{5}{2}R \Leftrightarrow s\tan\frac{A}{2} < \frac{5}{2}R \Leftrightarrow \frac{s}{4R}.\tan\frac{A}{2} < \frac{5}{8} \Leftrightarrow \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\tan\frac{A}{2} < \frac{5}{8} \Leftrightarrow$$

$$\Leftrightarrow \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} < \frac{5}{8}.$$

Let $x := \cos\frac{\pi - A}{2} \in \left(0, \frac{\sqrt{2}}{2}\right)$. By AM − GM and Jensen Inequalities, we have

$$\cos\frac{B}{2}\cos\frac{C}{2} \le \left(\frac{\cos\frac{B}{2} + \cos\frac{C}{2}}{2}\right)^2 = \cos^2\frac{B+C}{4} = \frac{1}{2}\left(1 + \cos\frac{B+C}{2}\right)$$

$$\Rightarrow \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \cos\frac{\pi - A}{2}.\frac{1}{2}\left(1 + \cos\frac{\pi - A}{2}\right) = \frac{x(1+x)}{2} \le \frac{\sqrt{2}}{4}\left(1 + \frac{\sqrt{2}}{2}\right) = \frac{2\sqrt{2}+2}{8} < \frac{5}{8}.$$