In any Δ ABC, the following relationship holds :

$$\frac{m_a w_a}{r_a h_a} \geq \frac{r}{R - r + \sqrt{R(R-2r)}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } p \to \text{semi}-\text{perimeter}, \frac{m_a w_a}{r_a h_a} \overset{\text{Lascu + A-G}}{\geq} \frac{pa(p-a)^2}{2r^2p^2}$$

$$= \frac{a(p-a)^2}{2(p-a)(p-b)(p-c)} = \frac{a}{2p}\cdot\frac{p(p-a)}{(p-b)(p-c)} = \frac{a}{2p}.\cot^2\frac{A}{2}$$

$$\overset{?}{\geq} \frac{r}{R-r+\sqrt{R(R-2r)}} \Leftrightarrow R - r + \sqrt{R(R-2r)} \overset{?}{\geq} \frac{2pr}{a}.\tan^2\frac{A}{2}$$

$$\Leftrightarrow R - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + R.\sqrt{1-4sc+c^2} \overset{?}{\geq}$$

$$\frac{2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}.\frac{s^2}{1-s^2}\left(s = \sin\frac{A}{2} \text{ and } c = \cos\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1 - 2s(c-s) + \sqrt{1-4sc+c^2} \overset{?}{\geq} 2(s+c)(c-s).\frac{s^2}{1-s^2}$$

$$\Leftrightarrow \sqrt{1-4sc+c^2} \overset{?}{\geq} 2(s+c)(c-s).\frac{s^2}{1-s^2} - 1 + 2s(c-s)$$

$$\Leftrightarrow \boxed{\sqrt{1-4sc+c^2} \underset{(*)}{\overset{?}{\geq}} \frac{2s^2c^2 + 2c(s-s^3) - (s^2+1)}{1-s^2}}$$

We note that for an equilateral triangle, $c = 1$ and $s = \frac{1}{2}$ and then : LHS of $(*)$ = RHS of $(*) = 0$ and moreover, if LHS of $(*) \leq 0$, then $(*)$ is evidently true and so, we now consider the case when : $2s^2c^2 + 2c(s-s^3) - (s^2+1) > 0$ and then :

$$(*) \Leftrightarrow 1 - 4sc + c^2 \geq \frac{\left(2s^2c^2 + 2c(s-s^3) - (s^2+1)\right)^2}{(1-s^2)^2}$$

$$\Leftrightarrow -c^4s - 2c^3(1-s^2) + c^2(3s-s^3) + 2c(1-s^2) + s^3 - 2s \geq 0$$

$$\Leftrightarrow -c^4s + (2c-2c^3)(1-s^2) + s^3(1-c^2) + s(3c^2-2) \geq 0$$

$$\Leftrightarrow s(3c^2 - 2 - c^4) + (1-c^2)\left(s^3 + 2c(1-s^2)\right) \geq 0$$

$$\Leftrightarrow s(1-c^2)(c^2-2) + (1-c^2)\left(s^3 + 2c(1-s^2)\right) \geq 0$$

$$\Leftrightarrow \boxed{(1-c^2)\left(sc^2 - 2s + s^3 + 2c(1-s^2)\right) \overset{(**)}{\geq} 0}$$

We have : $c = \cos\frac{B-C}{2} = \frac{b+c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2} = s \Rightarrow c > s$ and so, LHS of $(**) >$

$$(1-c^2)\left(s^3-2s+s^3+2s(1-s^2)\right)\ \left(\because 1-c^2=1-\cos^2\frac{B-C}{2}>0\right)$$

$$=(1-c^2).0 \Rightarrow \text{LHS of } (**)>0 \Rightarrow (**) \Rightarrow (*) \text{ is true (strict inequality)}$$

and hence, combining both scenarios, $(*)$ is true $\forall\ \Delta\ ABC$

$$\therefore \frac{m_a w_a}{r_a h_a} \geq \frac{r}{R-r+\sqrt{R(R-2r)}}\ \forall\ \Delta\ ABC, '' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$