ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a}{r_a+r_b+r_c} > \frac{r}{2R}$$

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$$2\sum ab-\sum a^2=$$
$$=2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr)=$$
$$=4r(4R+r)\ (1) a^2+(b+c)^2 \overset{AM-GM}{\geq} 2a(b+c)\ (2)$$
$$\text{we need to show}$$
$$\frac{h_a}{r_a+r_b+r_c} > \frac{r}{2R}\ \ or,$$
$$\frac{bc}{(4R+r)2R} > \frac{r}{2R}\ \ or, bc > r(4R+r)\ or,$$
$$4\,bc > 4r(4R+r)\ or,$$
$$4bc \overset{(1)}{>} 2\sum ab-\sum a^2$$
$$or, \sum a^2+4bc > 2\sum ab\ \ or,$$
$$\sum a^2+2bc > 2ab+2ac\ or,$$
$$(b^2+2bc+c^2)+a^2 > 2a(b+c)\ \ or,$$

$$a^2+(b+c)^2 > 2a(b+c)\ true\ \big(using\ (2)\big)$$