ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{w_a}{h_a\cos A}\le\frac{3R^2}{10r^2-2R^2}$$

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$$WLOG\ a\ge b\ge c\ \ then\ h_a\le h_b\le h_c\ and$$
$$\cos A\le\cos B\le\cos C\ \ (acute)\ and\ h_a\cos A\le h_b\cos B\le h_c\cos C$$

$$\sum w_a\le\sum\sqrt{s(s-a)}\overset{CBS}{\le}\sqrt{3s(3s-a-b-c)}=s\sqrt{3}\overset{Mitrinovic}{\le}$$
$$\le\ 3\sqrt{3}\,\frac{R}{2}\sqrt{3}=\frac{9R}{2}\ \ (1)$$
$$.\sum\tan A=\frac{2sr}{s^2-(2R+r)^2}=\frac{2F}{s^2-(2R+r)^2}\le Walker's$$
$$\le\frac{2F}{2R^2+8Rr+3r^2-(2R+r)^2}=$$
$$=\frac{2F}{4Rr+2r^2-2R^2}\overset{Euler}{\le}\frac{2F}{4.2r.r+2r^2-2R^2}\le\frac{2F}{10r^2-2R^2}\ \ (2)$$

$$\sum_{cyc}\frac{w_a}{h_a\cos A}\le\sum\frac{\sqrt{s(s-a)}}{h_a\cos A}\overset{Chebyshev}{\le}\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{1}{h_a\cos A}\right)=$$

$$=\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{a}{2F\cos A}\right)=\frac{1}{3}\left(\sum w_a\right)\left(\sum\frac{2R\sin A}{2\,F\cos A}\right)=$$

$$=\frac{R}{3F}\left(\sum w_a\right)\left(\sum\tan A\right)\overset{(1)\&(2)}{\le}\frac{R}{3F}\,\frac{9R}{2}\,\frac{2F}{10r^2-2R^2}=\frac{3R^2}{10r^2-2R^2}$$

$$Equality\ holds\ \ for\ an\ \ equilateral\ \ triangle$$