ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\sqrt{r_a+r_b}}{c}\right)^n \geq \frac{3}{R^{\frac{n}{2}}} \qquad n\in\mathbb{N}$$

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Solution by Tapas Das-India

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=\sum r_a\sum r_ar_b - r_ar_br_c=(4R+r)s^2-s^2r=4Rs^2 \ (1)$$

$$a^2b^2c^2=16R^2r^2s^2 \overset{Euler}{\leq} 16R^2\left(\frac{R}{2}\right)^2 s^2=4R^4s^2 \quad (2)$$

$$\sum_{cyc}\left(\frac{\sqrt{r_a+r_b}}{c}\right)^n=\sum_{cyc}\left(\sqrt{\frac{r_a+r_b}{c^2}}\right)^n \overset{AM-GM}{\geq}$$

$$\geq 3\left(\sqrt[3]{\sqrt{\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{a^2b^2c^2}}}\right)^n \overset{(1)\&(2)}{\geq} 3\left(\sqrt[3]{\sqrt{\frac{4Rs^2}{4R^4s^2}}}\right)^{\frac{n}{2}}=\frac{3}{R^{\frac{n}{2}}}$$

Equality holds for an equilateral triangle