ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sqrt[r_a]{a}\cdot\sqrt[r_b]{b}\cdot\sqrt[r_c]{c}\le\sqrt[r]{\frac{a+b+c}{3}}\le\sqrt[h_a]{a}\cdot\sqrt[h_b]{b}\cdot\sqrt[h_c]{c}$$

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$$WLOG\ a\ge b\ge c\ then\ r_a\ge r_b\ge r_c.$$

$$Now\ \frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\overset{Chebyshev}{\le}\frac{1}{3}(a+b+c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\ (1)$$

$$We\ know\ that\ \sum\frac{1}{r_a}=\sum\frac{1}{h_a}=\frac{1}{r}\ (2)$$

$$ah_a=bh_b=ch_c=a\frac{2F}{a}=b\frac{2F}{b}=c\frac{2F}{c}=2F\ (3)$$

Let us consider a with associated weight $\frac{1}{r_a}$, *b with associated weight* $\frac{1}{r_b}$, *c with associated weight* $\frac{1}{r_c}$

$$GM\le AM\ or\ \left(\sqrt[r_a]{a}.\sqrt[r_b]{b}.\sqrt[r_c]{c}\right)^{\frac{1}{\sum\frac{1}{r_a}}}\le\frac{\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}}{\sum\frac{1}{r_a}}\overset{(1)}{\le}\frac{\frac{1}{3}(a+b+c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}{\sum\frac{1}{r_a}}=$$

$$=\frac{a+b+c}{3}\ or\ \sqrt[r_a]{a}.\sqrt[r_b]{b}.\sqrt[r_c]{c}\le\left(\frac{a+b+c}{3}\right)^{\sum\frac{1}{r_a}}\overset{(2)}{=}$$

$$=\left(\frac{a+b+c}{3}\right)^{\frac{1}{r}}=\sqrt[r]{\frac{a+b+c}{3}}\ (A)$$

Let us consider a with associated weight $\frac{1}{h_a}$, *b with associated weight* $\frac{1}{h_b}$, *c with associated weight* $\frac{1}{h_c}$

$$GM\ge HM\ or\ \left(\sqrt[h_a]{a}\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{\frac{1}{\sum\frac{1}{h_a}}}\ge\frac{\sum\frac{1}{h_a}}{\sum\frac{1}{ah_a}}\overset{(3)}{\ge}\frac{\sum\frac{1}{h_a}}{\frac{3}{2F}}\overset{(2)}{=}\frac{2F}{3r}=2r.\frac{s}{3r}=\frac{2s}{3}=\frac{a+b+c}{3}$$

$$or\ \left(\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{\frac{1}{\sum\frac{1}{h_a}}}=\left(\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{\frac{1}{\frac{1}{r}}}=$$

$$=\left(\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{r}\geq\frac{a+b+c}{3}\ \ or$$

$$\sqrt[h_a]{a}\sqrt[h_b]{b}\sqrt[h_c]{c}\geq\sqrt[r]{\frac{a+b+c}{3}}\,(B)$$

$$From\ (A)\ and\ (B) we\ \ have\ \sqrt[r_a]{a}.\sqrt[r_b]{b}.\sqrt[r_c]{c}\leq\sqrt[r]{\frac{a+b+c}{3}}\leq\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}$$

$$Equality\ holds\ for\ the\ equilateral\ triangle.$$