ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, holds :

$$r_a^3 + r_b^3 + r_c^3 - 3r_ar_br_c \geq \frac{\sqrt{3}}{2}(a^3 + b^3 + c^3 - 3abc)$$

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$$r_a^3 + r_b^3 + r_c^3 - 3r_ar_br_c \geq \frac{\sqrt{3}}{2}(a^3 + b^3 + c^3 - 3abc)$$

$$\Leftrightarrow \left(\sum_{cyc} r_a\right)^3 - 3\left(\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_ar_b\right) - r_ar_br_c\right) - 3r_ar_br_c \geq$$

$$\frac{\sqrt{3}}{2}.2s(s^2 - 6Rr - 3r^2 - 6Rr)$$

$$\Leftrightarrow (4R + r)^3 - 3s^2(4R + r) \geq \sqrt{3}s(s^2 - 12Rr - 3r^2)$$

$$\Leftrightarrow \frac{(4R + r)}{\sqrt{3}s}((4R + r)^2 - 3s^2) \overset{(*)}{\geq} s^2 - 12Rr - 3r^2$$

Now, via Doucet (or Trucht), LHS of $(*) \geq 16R^2 + 8Rr + r^2 - 3s^2$

$$\overset{?}{\geq} s^2 - 12Rr - 3r^2 \Leftrightarrow s^2 \overset{?}{\leq} 4R^2 + 5Rr + r^2$$

$$\Leftrightarrow (s^2 - 4R^2 - 4Rr - 3r^2) - r(R - 2r) \overset{?}{\leq} 0$$

$$\rightarrow \text{true} \because s^2 \overset{\text{Gerretsen}}{\leq} 4R^2 + 4Rr + 3r^2 \text{ and } -r(R - 2r) \overset{\text{Euler}}{\leq} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore r_a^3 + r_b^3 + r_c^3 - 3r_ar_br_c \geq \frac{\sqrt{3}}{2}(a^3 + b^3 + c^3 - 3abc)$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)