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In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\right)\le\sqrt{3\left(1-\frac{r}{2R}\right)}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\right)^{\sin\frac{B}{2}}=\left(1+\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}-1\right)\right)^{\sin\frac{B}{2}}\overset{\text{Bernoulli}}{\le}1+\left(\sin\frac{B}{2}\right)\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}-1\right)$$

$$=1+\frac{\sin^2\frac{A}{2}}{\sin\frac{B}{2}}-\sin\frac{B}{2}\Rightarrow\left(\sin\frac{B}{2}\right)\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\right)^{\sin\frac{B}{2}}\le\sin\frac{B}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}$$

$$\Rightarrow\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\le\sin\frac{B}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\text{ and analogs}$$

$$\therefore\sum_{cyc}\left(\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\right)\le\sum_{cyc}\sin\frac{B}{2}+\sum_{cyc}\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)=$$

$$=\sum_{cyc}\sin\frac{A}{2}\overset{\text{Jensen}}{\le}\frac{3}{2}=\sqrt{\frac{9}{4}}=\sqrt{3-\frac{3r}{4r}}\overset{\text{Euler}}{\le}\sqrt{3-\frac{3r}{2R}}$$

$$\therefore\sum_{cyc}\left(\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\right)\le\sqrt{3\left(1-\frac{r}{2R}\right)}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)