ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}+\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}$$
$$+\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}}\le 1$$

*Proposed by Dang Ngoc Minh-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Weighted GM} \ge \text{Weighted HM} \Rightarrow \sqrt[\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)]{\left(\tan\frac{A}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_c}}}$$
$$\ge \frac{\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}}{\frac{r}{r_a.\tan\frac{A}{2}}+\frac{r}{r_b.\tan\frac{B}{2}}+\frac{r}{r_c.\tan\frac{C}{2}}} \text{ and } \because \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1,$$
$$\therefore \left(\tan\frac{A}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_c}} \ge \frac{1}{rs.\sum_{cyc}\frac{1}{r_a^2}} = \frac{1}{rs.\left(\left(\sum_{cyc}\frac{1}{r_a}\right)^2-2\sum_{cyc}\frac{1}{r_ar_b}\right)}$$
$$=\frac{1}{rs.\left(\frac{1}{r^2}-\frac{2(4R+r)}{rs^2}\right)}=\frac{1}{rs.\left(\frac{s^2-8Rr-2r^2}{r^2s^2}\right)}=\frac{rs}{s^2-8Rr-2r^2}$$
$$\therefore \left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}=\frac{\left(\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}\right)\left(\tan\frac{C}{2}\right)}{\left(\tan\frac{A}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_c}}}$$
$$\boxed{\le}\ \frac{\left(\frac{r}{s}\right)}{\left(\frac{rs}{s^2-8Rr-2r^2}\right)}=\frac{s^2-8Rr-2r^2}{s^2}\rightarrow(1)$$

$$\text{Again, Weighted GM} \ge \text{Weighted HM} \Rightarrow \sqrt[\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)]{\left(\tan\frac{B}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_c}}}$$
$$\ge \frac{\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}}{\frac{r}{r_a.\tan\frac{B}{2}}+\frac{r}{r_b.\tan\frac{C}{2}}+\frac{r}{r_c.\tan\frac{A}{2}}} \text{ and } \because \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1,$$
$$\therefore \left(\tan\frac{B}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_c}} \ge \frac{1}{rs.\sum_{cyc}\frac{1}{r_ar_b}}=\frac{1}{\frac{rs}{rs^2}(4R+r)}=\frac{s}{4R+r}$$
$$\therefore \left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}=\frac{\left(\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}\right)\left(\tan\frac{C}{2}\right)}{\left(\tan\frac{B}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_c}}}$$

$$\boxed{\leq}\frac{\left(\frac{r}{s}\right)}{\left(\frac{s}{4R+r}\right)}=\frac{4Rr+r^2}{s^2}\to(2)$$

Also, Weighted GM $\geq$ Weighted HM $\Rightarrow$ $\sqrt[\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)]{\left(\tan\frac{C}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_c}}}$

$$\geq\frac{\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}}{\frac{r}{r_a.\tan\frac{C}{2}}+\frac{r}{r_b.\tan\frac{A}{2}}+\frac{r}{r_c.\tan\frac{B}{2}}}\ \textit{and}\ \because\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1,$$

$$\therefore\left(\tan\frac{C}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_c}}\geq\frac{1}{rs.\sum_{cyc}\frac{1}{r_ar_c}}=\frac{1}{\frac{rs}{rs^2}(4R+r)}=\frac{s}{4R+r}$$

$$\therefore\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}}=\frac{\left(\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}\right)\left(\tan\frac{C}{2}\right)}{\left(\tan\frac{C}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_c}}}$$

$$\boxed{\leq}\frac{\left(\frac{r}{s}\right)}{\left(\frac{s}{4R+r}\right)}=\frac{4Rr+r^2}{s^2}\to(3)\ \therefore(1)+(2)+(3)\Rightarrow$$

$$\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}+\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}$$

$$+\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}}\leq\frac{s^2-8Rr-2r^2}{s^2}+2.\frac{4Rr+r^2}{s^2}$$

$$\therefore\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}+\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}$$

$$+\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}}\leq 1$$

$\forall\,\Delta\,ABC$, " $=$ " iff $\Delta\,ABC$ is equilateral (QED)