ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$2+\sum_{cyc}\frac{r_a r_b}{r_c(r_a+r_b)}\le\frac{7}{16}\prod_{cyc}\left(1+\frac{r_a}{r_b}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2+\sum_{cyc}\frac{r_a r_b}{r_c(r_a+r_b)}-\frac{7}{16}\prod_{cyc}\left(1+\frac{r_a}{r_b}\right)=$$

$$=2+\sum_{cyc}\frac{s(s-c)}{\frac{rs}{s-c}.4R.\frac{s(s-c)}{ab}}-\frac{7}{16}.\prod_{cyc}\frac{4R\cos^2\frac{C}{2}}{r_b}=2+\sum_{cyc}\frac{s-c}{\left(\frac{abc}{ab}\right)}-\frac{7}{16}.\frac{64R^3.\frac{s^2}{16R^2}}{rs^2}=$$

$$=2+\frac{s(s^2+4Rr+r^2)}{4Rrs}-3-\frac{7R}{4r}=\frac{s^2+r^2}{4Rr}-\frac{7R}{4r}=\frac{s^2+r^2-7R^2}{4Rr}\overset{\text{Gerretsen}}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2+r^2-7R^2}{4Rr}=-\frac{3R^2-4Rr-4r^2}{4Rr}=-\frac{(R-2r)(3R+2r)}{4Rr}\overset{\text{Euler}}{\le}0$$

$$\therefore 2+\sum_{cyc}\frac{r_a r_b}{r_c(r_a+r_b)}\le\frac{7}{16}\prod_{cyc}\left(1+\frac{r_a}{r_b}\right)$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)