ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC, the following relationship holds :

$$\frac{r_a r_b a^2 b^2}{(r_a+r_b)m_a^2 m_b^2}+\frac{r_b r_c b^2 c^2}{(r_b+r_c)m_b^2 m_c^2}+\frac{r_c r_a c^2 a^2}{(r_c+r_a)m_c^2 m_a^2}\ge 8r$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\frac{m_a m_b}{ab}=\frac{9}{4}\sum_{cyc}\frac{AG.BG}{ab}\overset{Hayashi}{\ge}\frac{9}{4}$$

and implementing : $\sum_{cyc}\frac{m_a m_b}{ab}\ge\frac{9}{4}$ on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$

whose medians as a consequence of Apollonius' theorem

$=\frac{a}{2},\frac{b}{2},\frac{c}{2}$ respectively, we arrive at :

$$\sum_{cyc}\frac{\frac{1}{4}.ab}{\frac{4}{9}.m_a m_b}\ge\frac{9}{4}\Rightarrow\sum_{cyc}\frac{ab}{m_a m_b}\ge 4\rightarrow(1)$$

$$\text{Now, }\sum_{cyc}\frac{r_a r_b a^2 b^2}{(r_a+r_b)m_a^2 m_b^2}=\sum_{cyc}\frac{\left(\frac{ab}{m_a m_b}\right)^2}{\frac{1}{r_a}+\frac{1}{r_b}}\overset{Bergstrom}{\ge}$$

$$\ge\frac{\left(\sum_{cyc}\frac{ab}{m_a m_b}\right)^2}{2\sum_{cyc}\frac{1}{r_a}}\overset{via\ (1)}{\ge}\frac{16}{\frac{2}{r}}=8r$$

$$\therefore\sum_{cyc}\frac{r_a r_b a^2 b^2}{(r_a+r_b)m_a^2 m_b^2}\ge 8r\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$