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In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \tan^2\frac{A}{2} \ge \frac{26}{27} + 27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3}$$

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$$\sum_{cyc}\tan^2\frac{A}{2} = \frac{1}{s^2}\sum_{cyc} r_a^2 = \frac{(4R+r)^2 - 2s^2}{s^2} \overset{\text{Trucht}}{\ge} \frac{3s^2 - 2s^2}{s^2}$$

$$\Rightarrow \sum_{cyc}\tan^2\frac{A}{2} \ge 1 \to (1)$$

$$\text{Again, } \sum_{cyc}\cot^2\frac{A}{2} = \sum_{cyc}\frac{s^2}{r_a^2} = \frac{s^2}{r^2s^4}\cdot\sum_{cyc} r_b^2 r_c^2 = \frac{1}{r^2s^2}\cdot\left(\left(\sum_{cyc} r_b r_c\right)^2 - 2r_a r_b r_c \sum_{cyc} r_a\right)$$

$$= \frac{s^4 - 2rs^2(4R+r)}{r^2s^2} = \frac{s^2 - 8Rr - 2r^2}{r^2} \Rightarrow \frac{26}{27} + 27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3}$$

$$= \frac{26}{27} + \frac{27r^6}{(s^2 - 8Rr - 2r^2)^3} \overset{?}{\le} 1 \Leftrightarrow \frac{729r^6}{(s^2 - 8Rr - 2r^2)^3} \overset{?}{\le} 1 \Leftrightarrow 9r^2 \overset{?}{\le} s^2 - 8Rr - 2r^2$$

$$\Leftrightarrow s^2 - 8Rr - 11r^2 \overset{?}{\ge} 0 \Leftrightarrow s^2 - 16Rr + 5r^2 + 8r(R - 2r) \overset{?}{\ge} 0$$

$$\to \text{true} \because s^2 - 16Rr + 5r^2 \overset{\text{Gerretsen}}{\ge} 0 \text{ and } 8r(R-2r) \overset{\text{Euler}}{\ge} 0$$

$$\therefore \frac{26}{27} + 27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3} \le 1 \overset{\text{via (1)}}{\le} \sum_{cyc}\tan^2\frac{A}{2} \Rightarrow$$

$$\sum_{cyc}\tan^2\frac{A}{2} \ge \frac{26}{27} + 27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3} \quad \forall \Delta \text{ ABC}, ''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$