ROMANIAN MATHEMATICAL MAGAZINE

If in $\Delta ABC, a \neq b \neq c \neq a$ then:

$$\left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 < 25R^2 + \frac{8\sqrt{3}\,F}{9}$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das-India

$$a^3(b-c)+b^3(c-a)+c^3(a-b) = a^3(b-c)+b^3c-c^3b+b^3a+c^3a$$
$$= a^2(b-c)+bc(b^2-c^2)-a(b^3-c^3)$$
$$= a^3(b-c)+bc(b+c)(b-c)-a(b-c)(b^2+bc+c^2)$$
$$= (b-c)\{a^3+b^2c+bc^2-ab^2-abc-ac^2\}$$
$$= (b-c)\{a(a^2-b^2)+bc(b-a)+c^2(b-a)\}$$
$$= (b-c)(a-b)\{a(a+b)-bc-c^2\} = (b-c)(a-b)\{a^2+ab-bc-c^2\}$$
$$= (b-c)(a-b)\{(a-c)(a+c)+b(a-c)\} = -(a-b)(b-c)(c-a)(a+b+c)$$
$$\therefore a^3(c-b)+b^3(a-c)+c^3(b-a) = -\{-(a-b)(b-c)(c-a)(a+b+c)\}$$
$$= (a-b)(b-c)(c-a)(a+b+c) \quad (1)$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$
$$= a^2b-a^2c+b^2c-b^2a+ac^2-bc^2+abc-abc$$
$$= ab(a-b+c)-ac(a-c)+bc(-a+b-c)$$
$$= (ab-bc)(a-b+c)-ac(a-c) = (a-c)\{b(a-b)+c(b-a)\}$$
$$= (a-c)(a-b)(b-c) = -(a-b)(b-c)(c-a)$$
$$\therefore a^2(c-b)+b^2(a-c)+c^2(b-a)$$
$$= -\{-(a-b)(b-c)(c-a)\} = (a-b)(b-c)(c-a) \quad (2)$$
$$\therefore \left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 = (a+b+c)^2 = 4s^2 \text{ (using (1) and (2))}$$

We need to show:

$$4s^2 < 25R^2 + \frac{8\sqrt{3}F}{9}$$

or $4(4R^2+4Rr+3r^2) < 25R^2 + \frac{8\sqrt{3}}{9}\cdot r\cdot 3\sqrt{3}r$

(Gerretsen an Mitrinovic)

or $9R^2 - 16Rr - 4r^2 > 0$

or $(R - 2r)(9R + 2r) > 0$ True (Euler)

Conclusion:

$$\left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 = 4s^2 < 25R^2 + \frac{8\sqrt{3}F}{9}$$

or, $\left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 < 25R^2 + \frac{8\sqrt{3}F}{9}$