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In any Δ ABC, the following relationship holds :

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq 2s$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Rovsen Pirguliyev-Azerbaijan

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq 2s \quad (*)$$

To prove that :

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq a+b+c \quad (1)$$

$$\left(\frac{a^2+bc}{b+c}-a\right)+\left(\frac{b^2+ca}{c+a}-b\right)+\left(\frac{c^2+ab}{a+b}-c\right)\geq 0$$

$$\frac{a^2+bc-ab-ca}{b+c}+\frac{b^2+ca-bc-ab}{c+a}+\frac{c^2+ab-ac-bc}{a+b}\geq 0$$

or $(a-b)\cdot\left(\frac{a-c}{b+c}-\frac{b-c}{c+a}\right)+(b-c)\cdot\left(\frac{b-a}{c+a}-\frac{c-a}{a+b}\right)+(c-a)\cdot\left(\frac{c-b}{a+b}-\frac{a-b}{b+c}\right)\geq 0$ or

$$(a-b)^2\cdot\frac{a+b}{(b+c)(c+a)}+(b-c)^2\cdot\frac{b+c}{(c+a)(a+b)}+\frac{(c-a)^2\cdot(c+a)}{(a+b)(b+c)}\geq 0$$

$a+b+c=2s$ then true (*)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^2b+\sum_{cyc}ab^2=\sum_{cyc}\left(ab\left(\sum_{cyc}a-c\right)\right)=\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc$$

$$=\sum_{cyc}a^2b+\sum_{cyc}ab^2\to(1)$$

Now, $\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}=\sum_{cyc}\frac{a^4}{a^2b+a^2c}+\sum_{cyc}\frac{b^2c^2}{b^2c+bc^2}\overset{\text{Bergstrom}}{\geq}$

$$\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}+\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}\overset{A-G}{\geq}\frac{2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)}{\sum_{cyc}a^2b+\sum_{cyc}ab^2}$$

$$\overset{\text{via }(1)}{=}\frac{2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)}{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc}=\frac{4(s^2-4Rr-r^2)(s^2+4Rr+r^2)}{2s(s^2+4Rr+r^2)-12Rrs}$$

$$=\frac{2(s^4-(4Rr+r^2)^2)}{s(s^2-2Rr+r^2)}\overset{?}{\geq}2s\Leftrightarrow s^4-(4Rr+r^2)^2\overset{?}{\geq}s^4-s^2(2Rr-r^2)$$

$$\Leftrightarrow (2R-r)s^2 \underbrace{\overset{?}{\geq}}_{(*)} r(4R+r)^2$$

$$\text{Now, } (2R-r)s^2 \overset{\text{Gerretsen}}{\geq} (2R-r)(16Rr-5r^2) \overset{?}{\geq} r(4R+r)^2$$

$$\Leftrightarrow 8R^2-17Rr+2r^2 \overset{?}{\geq} 0 \Leftrightarrow (8R-r)(R-2r) \overset{?}{\geq} 0 \to \text{true via Euler} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq 2s\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 3 by Şerban George Florin-Romania

$$\sum_{cyc}\frac{a^2+bc}{b+c} \geq 2s, \sum_{cyc}\left(\frac{a^2+bc}{b+c}+a\right) \geq 2s+\sum_{cyc}a,$$

$$\sum_{cyc}\frac{a^2+bc+ab+ac}{b+c} \geq 2\sum_{cyc}a,$$

$$\sum_{cyc}\frac{(a+b)(a+c)}{b+c} \geq 2\sum_{cyc}a, a+b=z, b+c=x, a+c=y,$$

$$\sum_{cyc}\frac{yz}{x} \geq \sum_{cyc}x,$$

$$\sum_{cyc}x=p, \sum_{cyc}xy=q, r=\prod_{cyc}x, \sum_{cyc}\frac{yz}{x}=\frac{1}{xyz}\sum_{cyc}(yz)^2 \geq \sum_{cyc}x,$$

$$\frac{q^2-2pr}{r} \geq p, q^2-2pr \geq pr,$$

$q^2 \geq 3pr$, true, Schur inequality. Then :

$$\sum_{cyc}\frac{a^2+bc}{b+c} \geq 2s$$

equality is if $a=b=c$.