ROMANIAN MATHEMATICAL MAGAZINE

If $x \in \mathbb{R}$ then in any ΔABC the following relationship holds:

$$a\sum\sqrt{(a\sin x)^2+(b\cos x)^2}\ge 2\sqrt{6}\cdot F$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\sum a\sqrt{a^2\sin^2 x+b^2\cos^2 x}\ge\sum a\frac{a|\sin x|+b|\cos x|}{\sqrt{2}}=$$

$$=\frac{1}{\sqrt{2}}\Big((a^2+b^2+c^2)|\sin x|+(ab+ac+bc)|\cos x|\Big)\ge$$

$$\ge\frac{1}{\sqrt{2}}(ab+ac+bc)(|\sin x|+|\cos x|)$$

$$\ge\frac{1}{\sqrt{2}}\cdot 4\sqrt{3}F\cdot 1=\frac{4\sqrt{6}F}{2}=2\sqrt{6}F$$

$$f(x)=\sin x+\cos x\quad x\in\left(0;\frac{\pi}{2}\right)$$

$$f'(x)=\cos x-\sin x=0\Rightarrow x=\frac{\pi}{4}$$

0	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f'(x)$	$+++++++++++++$	0	$-------------------$
$f(x)$	$1\ \nearrow$	$\sqrt{2}$	$\searrow\ 1$

$$\Rightarrow f(x)\ge 1$$

Solution 2 by Tapas Das-India

$$\sum a\sqrt{(a\sin x)^2+(b\cos x)^2}=\sum\sqrt{(a^2\sin x)^2+(ab\cos x)^2}\ge$$

$$\ge\sqrt{\left(|\sin x|\sum a^2\right)^2+\left(|\cos x|\sum ab\right)^2}\ (Minkowski)\ge$$

$$\ge\frac{1}{\sqrt{2}}\left[|\sin x|\sum a^2+|\cos x|\sum ab\right]\ge$$

$$\geq \frac{1}{\sqrt{2}}\left[|\sin x|\sum ab + |\cos x|\sum ab\right] =$$

$$= \frac{1}{\sqrt{2}}\sum ab\,(|\sin x| + |\cos x|) \overset{Gordon}{\geq} 4\sqrt{3}\frac{F}{\sqrt{2}} = 2\sqrt{6}\,F$$

$$\boldsymbol{Note}: (|\sin x| + |\cos x|)^2 = 1 + |\sin 2x| \geq 1 + 0 = 1.$$

$$|\sin x| + |\cos x| \geq 1$$