ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with

K → Lemoine point, the following relationship holds :

$$\frac{[BKC]}{ar_a}\cdot\sqrt{1+\frac{2[BKC]}{ar_a}}+\frac{[CKA]}{br_b}\cdot\sqrt{1+\frac{2[CKA]}{br_b}}+\frac{[AKB]}{cr_c}\cdot\sqrt{1+\frac{[AKB]}{cr_c}}\le\frac{\sqrt{3}}{3}$$

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$$\text{Let } AK_A = s_a \therefore \frac{KK_A}{AK} \overset{\text{Honsberger}}{=} \frac{a^2}{b^2+c^2} \Rightarrow \frac{KK_A}{AK}+1=\frac{a^2}{b^2+c^2}+1$$

$$\Rightarrow \frac{s_a}{AK}=\frac{a^2+b^2+c^2}{b^2+c^2} \Rightarrow AK=\frac{b^2+c^2}{a^2+b^2+c^2}.s_a=\frac{b^2+c^2}{a^2+b^2+c^2}.\frac{2bc}{b^2+c^2}.m_a$$

$$\Rightarrow AK=\frac{2bc}{\sum_{cyc}a^2}.m_a \text{ and analogs} \rightarrow (1)$$

$$\text{Now, } \Delta\, BKC \text{ has sides } BK, CK \text{ and } 'a' \therefore 16[BKC]^2$$

$$=2a^2BK^2+2a^2CK^2+2BK^2CK^2-a^4-BK^4-CK^4$$

$$=2a^2.\frac{4c^2a^2}{\left(\sum_{cyc}a^2\right)^2}.m_b^2+2a^2.\frac{4a^2b^2}{\left(\sum_{cyc}a^2\right)^2}.m_c^2+2.\frac{4c^2a^2}{\left(\sum_{cyc}a^2\right)^2}.m_b^2.\frac{4a^2b^2}{\left(\sum_{cyc}a^2\right)^2}.m_c^2$$

$$-a^4-\frac{16c^4a^4}{\left(\sum_{cyc}a^2\right)^4}.m_b^4-\frac{16a^4b^4}{\left(\sum_{cyc}a^2\right)^4}.m_c^4=$$

$$\frac{1}{\left(\sum_{cyc}a^2\right)^4}\left(\begin{gathered}2a^2.\left(\sum_{cyc}a^2\right)^2.c^2a^2(2c^2+2a^2-b^2)+2a^2.\left(\sum_{cyc}a^2\right)^2.a^2b^2(2a^2+2b^2-c^2)\\ +2c^2a^2.a^2b^2(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)-a^4\left(\sum_{cyc}a^2\right)^4\\ -c^4a^4(2c^2+2a^2-b^2)^2-a^4b^4(2a^2+2b^2-c^2)^2\end{gathered}\right)$$

$\rightarrow (2)$

$$\text{Now, } 2c^2a^2.a^2b^2(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)-c^4a^4(2c^2+2a^2-b^2)^2$$

$$-a^4b^4(2a^2+2b^2-c^2)^2=-\left(c^2a^2(2c^2+2a^2-b^2)-a^2b^2(2a^2+2b^2-c^2)\right)^2$$

$$=-a^4\left(2(c^2+b^2)(c^2-b^2)+2a^2(c^2-b^2)\right)^2=-4a^4(c^2-b^2)^2\left(\sum_{cyc}a^2\right)^2 \rightarrow (\text{i})$$

$$\text{Again, } 2a^2.\left(\sum_{cyc}a^2\right)^2.c^2a^2(2c^2+2a^2-b^2)+$$

$$2a^2.\left(\sum_{cyc}a^2\right)^2.a^2b^2(2a^2+2b^2-c^2)-a^4\left(\sum_{cyc}a^2\right)^4$$

$$= a^4\left(\sum_{cyc} a^2\right)^2\left(2c^2(2c^2+2a^2-b^2)+2b^2(2a^2+2b^2-c^2)-\left(\sum_{cyc} a^2\right)^2\right)$$

$$\rightarrow (ii) \therefore (2),(i),(ii) \Rightarrow 16[BKC]^2$$

$$= \frac{a^4}{\left(\sum_{cyc} a^2\right)^2}\cdot\begin{pmatrix} 2c^2(2c^2+2a^2-b^2)+2b^2(2a^2+2b^2-c^2) \\ -\left(\sum_{cyc} a^2\right)^2-4(c^2-b^2)^2\end{pmatrix}$$

$$= \frac{a^4}{\left(\sum_{cyc} a^2\right)^2}\cdot\left(2\sum_{cyc} a^2b^2-\sum_{cyc} a^4\right) = \frac{a^4}{\left(\sum_{cyc} a^2\right)^2}.16F^2 \Rightarrow [BKC] = \frac{a^2.F}{\sum_{cyc} a^2}$$

$$\Rightarrow \frac{[BKC]}{ar_a} = \frac{a^2F(s-a)}{\left(\sum_{cyc} a^2\right)aF} \Rightarrow \boxed{\frac{[BKC]}{ar_a} = \frac{a(s-a)}{\sum_{cyc} a^2}} \text{ and analogs} \rightarrow (*)$$

$$\text{Now, } \frac{[BKC]}{ar_a}.\sqrt{1+\frac{2[BKC]}{ar_a}}+\frac{[CKA]}{br_b}.\sqrt{1+\frac{2[CKA]}{br_b}}+\frac{[AKB]}{cr_c}.\sqrt{1+\frac{[AKB]}{cr_c}}$$

$$= \sum_{cyc} x.\sqrt{1+2x}\ \left(x = \frac{[BKC]}{ar_a} \text{ and analogs}\right)$$

$$= \sum_{cyc}\left(\sqrt{x}.\sqrt{x+2x^2}\right) \underset{(**)}{\overset{CBS}{\leq}} \sqrt{\sum_{cyc} x}.\sqrt{\sum_{cyc} x+2\sum_{cyc} x^2}$$

$$\sum_{cyc} x^2 \overset{\text{via } (*)}{=} \frac{\sum_{cyc} a^2(s-a)^2}{\left(\sum_{cyc} a^2\right)^2} = \frac{1}{\left(\sum_{cyc} a^2\right)^2}.\sum_{cyc} a^2(s^2-2sa+a^2)$$

$$= \frac{1}{\left(\sum_{cyc} a^2\right)^2}.\begin{pmatrix} 2s^2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+2(s^2+4Rr+r^2)^2 \\ -32Rrs^2-16r^2s^2\end{pmatrix}$$

$$= \frac{2r^2\left((4R+r)^2-s^2\right)}{\left(\sum_{cyc} a^2\right)^2} \Rightarrow \sum_{cyc} x+2\sum_{cyc} x^2 \overset{\text{via } (*)}{=}$$

$$\frac{\sum_{cyc} a(s-a)}{\sum_{cyc} a^2}+\frac{4r^2\left((4R+r)^2-s^2\right)}{\left(\sum_{cyc} a^2\right)^2} = \frac{4Rr+r^2}{s^2-4Rr-r^2}+\frac{r^2\left((4R+r)^2-s^2\right)}{(s^2-4Rr-r^2)^2}$$

$$= \frac{4Rrs^2}{(s^2-4Rr-r^2)^2} \therefore \text{via } (**),$$

$$\frac{[BKC]}{ar_a}.\sqrt{1+\frac{2[BKC]}{ar_a}}+\frac{[CKA]}{br_b}.\sqrt{1+\frac{2[CKA]}{br_b}}+\frac{[AKB]}{cr_c}.\sqrt{1+\frac{[AKB]}{cr_c}}$$

$$\leq \sqrt{\sum_{cyc}\frac{[BKC]}{ar_a}}.\sqrt{\frac{4Rrs^2}{(s^2-4Rr-r^2)^2}} \overset{\text{via } (*)}{=} \sqrt{\frac{a(s-a)}{\sum_{cyc} a^2}}.\sqrt{\frac{4Rrs^2}{(s^2-4Rr-r^2)^2}}$$

$$= \sqrt{\frac{4Rr+r^2}{s^2-4Rr-r^2}}.\sqrt{\frac{4Rrs^2}{(s^2-4Rr-r^2)^2}} \overset{?}{\leq} \frac{\sqrt{3}}{3}$$

$$\Leftrightarrow (s^2-4Rr-r^2)^3 \underset{(***)}{\overset{?}{\geq}} 12Rrs^2(4Rr+r^2)$$

$$\text{Firstly, } \left(\sum_{cyc} ab\right)^2 \ge 3abc\sum_{cyc} a = 12Rrs.2s \Rightarrow \left(\sum_{cyc} ab\right)^2 \ge 24Rrs^2$$

$$\Rightarrow \left(\sum_{cyc} a^2\right)^2 \ge 24Rrs^2 \Rightarrow \left(s^2 - 4Rr - r^2\right)^2 \overset{(\bullet)}{\ge} 6Rrs^2$$

$$\text{Also, } s^2 - 12Rr - 3r^2 = s^2 - 16Rr + 5r^2 + 4r(R - 2r) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\ge} 0$$

$$\Rightarrow s^2 - 4Rr - r^2 \overset{(\bullet\bullet)}{\ge} 8Rr + 2r^2 \therefore (\bullet).(\bullet\bullet) \Rightarrow (***) \text{ is true}$$

$$\therefore \frac{[BKC]}{ar_a}.\sqrt{1 + \frac{2[BKC]}{ar_a}} + \frac{[CKA]}{br_b}.\sqrt{1 + \frac{2[CKA]}{br_b}} + \frac{[AKB]}{cr_c}.\sqrt{1 + \frac{[AKB]}{cr_c}}$$

$$\le \frac{\sqrt{3}}{3}\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$