ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a^4}{(b+c)m_b^2m_c^2}+\frac{b^4}{(c+a)m_c^2m_a^2}+\frac{c^4}{(a+b)m_a^2m_b^2}\ge\frac{8\sqrt{3}}{9R}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2}{bc}\overset{?}{\ge}\frac{9}{4}\Leftrightarrow\frac{1}{16Rrs}\cdot\sum_{cyc}a(2b^2+2c^2-a^2)\overset{?}{\ge}\frac{9}{4}$$

$$\Leftrightarrow\frac{1}{4Rrs}\cdot\sum_{cyc}a(2b^2+2c^2+2a^2-3a^2)\overset{?}{\ge}9$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)-3\sum_{cyc}a^3\overset{?}{\ge}36Rrs$$

$$\Leftrightarrow 8s(s^2-4Rr-r^2)-6s(s^2-6Rr-3r^2)\overset{?}{\ge}36Rrs\Leftrightarrow s^2\overset{?}{\ge}16Rr-5r^2$$

$\rightarrow$ true via Gerretsen $\therefore \sum_{cyc}\frac{m_a^2}{bc}\ge\frac{9}{4}$ and implementing : $\sum_{cyc}\frac{m_a^2}{bc}\ge\frac{9}{4}$ on

a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose medians as a consequence of

Apollonius' theorem $=\frac{a}{2},\frac{b}{2},\frac{c}{2}$ respectively, we arrive at : $\sum_{cyc}\frac{\frac{1}{4}\cdot a^2}{\frac{4}{9}m_bm_c}\ge\frac{9}{4}$

$$\Rightarrow\sum_{cyc}\frac{a^2}{m_bm_c}\ge 4\rightarrow(1)$$

Now, $\sum_{cyc}\frac{a^4}{(b+c)m_b^2m_c^2}=\sum_{cyc}\frac{\left(\frac{a^2}{m_bm_c}\right)^2}{b+c}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}\frac{a^2}{m_bm_c}\right)^2}{4s}\overset{\substack{\text{via }(1)\\ \text{and via Mitrinovic}}}{\ge}\frac{16}{4R\cdot\frac{3\sqrt{3}}{2}}$

$$=\frac{8\sqrt{3}}{9R}\ \forall\,\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$\frac{a^4}{(b+c)m_b^2m_c^2}+\frac{b^4}{(c+a)m_c^2m_a^2}+\frac{c^4}{(a+b)m_a^2m_b^2}\overset{\text{Bergstrom}}{\ge}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{\sum(b+c)m_b^2m_c^2} \overset{Chebyshev}{\geq} \frac{\frac{16}{9}\left(\sum m_a^2\right)^2}{\frac{1}{3}(4s)\left(\sum m_b^2m_c^2\right)} \geq$$

$$= \frac{\frac{16}{9}3\left(\sum m_b^2m_c^2\right)}{\frac{1}{3}(4s)\left(\sum m_b^2m_c^2\right)} = \frac{12}{3s} \overset{Mitrinovic}{\geq} \frac{12}{9\sqrt{3}\frac{R}{2}} = \frac{8\sqrt{3}}{9R}$$

Equality holds for $a=b=c$.