ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a^4}{(b+c)b^2c^2}+\frac{m_b^4}{(c+a)c^2a^2}+\frac{m_c^4}{(a+b)a^2b^2}\geq\frac{9\sqrt{3}}{32R}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^4}{(b+c)b^2c^2}=\sum_{cyc}\frac{\left(\frac{m_a^2}{bc}\right)^2}{b+c}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{m_a^2}{bc}\right)^2}{4s}\overset{\text{Mitrinovic}}{\geq}$$

$$\geq\frac{\left(\sum_{cyc}\frac{m_a^2}{bc}\right)^2}{4R.\frac{3\sqrt{3}}{2}}\overset{?}{\geq}\frac{9\sqrt{3}}{32R}\Leftrightarrow\sum_{cyc}\frac{m_a^2}{bc}\overset{?}{\geq}\frac{9}{4}\Leftrightarrow\frac{1}{16Rrs}.\sum_{cyc}a(2b^2+2c^2-a^2)\overset{?}{\geq}\frac{9}{4}$$

$$\Leftrightarrow\frac{1}{4Rrs}.\sum_{cyc}a(2b^2+2c^2+2a^2-3a^2)\overset{?}{\geq}9$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)-3\sum_{cyc}a^3\overset{?}{\geq}36Rrs$$

$$\Leftrightarrow 8s(s^2-4Rr-r^2)-6s(s^2-6Rr-3r^2)\overset{?}{\geq}36Rrs\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2$$

$$\rightarrow\text{true via Gerretsen}\therefore\sum_{cyc}\frac{m_a^4}{(b+c)b^2c^2}\geq\frac{9\sqrt{3}}{32R}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)