ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$2R\sum_{cyc}\cos\frac{A-B}{2}\le\sum_{cyc}\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2=\frac{m_a^2}{\frac{2bc}{b^2+c^2}\cdot m_a\cos^2\frac{A}{2}}\overset{A-G}{\ge}\frac{m_a}{\cos^2\frac{A}{2}}\overset{Lascu}{\ge}\frac{\frac{b+c}{2}\cdot\cos\frac{A}{2}}{\cos^2\frac{A}{2}}$$

$$=\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2\cos\frac{A}{2}}\Rightarrow\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2\ge 2R\cos\frac{B-C}{2}\ \textit{and analogs}\Rightarrow$$

$$\sum_{cyc}\left(\frac{m_a}{\sqrt{s_a}\cos\frac{A}{2}}\right)^2\ge 2R\sum_{cyc}\cos\frac{A-B}{2}\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$4m_a^2=2b^2+2c^2-a^2=2bc\cos A+b^2+c^2,\ Now\ 4m_a^2\overset{AM-GM}{\ge}2bc(1+\cos A)\ or,$$

$$m_a\ge\sqrt{bc}\cos\frac{A}{2}\ (1),$$

$$\left(\frac{m_a}{\left(\sqrt{s_a}\cos\frac{A}{2}\right)}\right)^2=\frac{m_a^2(b^2+c^2)}{2bcm_a\cos^2\frac{A}{2}}=$$

$$=\frac{m_a(b^2+c^2)}{2\,bc\cos^2\frac{A}{2}}\overset{(1)\&\,CBS}{\ge}\frac{\sqrt{bc}\cos\frac{A}{2}(b+c)^2}{4\,bc\cos^2\frac{A}{2}}\overset{AM-GM}{\ge}$$

$$\ge\frac{b+c}{2}\frac{1}{\cos\frac{A}{2}}=\frac{R(\sin B+\sin C)}{\cos\frac{A}{2}}=\frac{2R\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{A}{2}}=2R\cos\frac{B-C}{2}$$

using this result we get

$$2R\sum\cos\left(\frac{A-B}{2}\right)\le\sum\left(\frac{m_a}{\left(\sqrt{s_a}\cos\frac{A}{2}\right)}\right)^2$$