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In ΔABC holds :

$$\frac{m_a}{\sqrt{m_b m_c}}+\frac{m_b}{\sqrt{m_c m_a}}+\frac{m_c}{\sqrt{m_a m_b}}\le 3\sqrt{\frac{R}{2r}}$$

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Let S be the area of ΔABC. By CBS inequality, we have

$$\frac{m_a}{\sqrt{m_b m_c}}+\frac{m_b}{\sqrt{m_c m_a}}+\frac{m_c}{\sqrt{m_a m_b}}\le\sqrt{({m_a}^2+{m_b}^2+{m_c}^2)\left(\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{1}{m_a m_b}\right)}.$$

So it suffices to prove that

$$({m_a}^2+{m_b}^2+{m_c}^2)\left(\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{1}{m_a m_b}\right)\le\frac{9R}{2r}=\frac{9abc(a+b+c)}{16S^2}.$$

We know that m_a, m_b, m_c can be the sides of a triangle with area $S' = \frac{3S}{4}$ and medians $m'_a = \frac{3a}{4}$, $m'_b = \frac{3b}{4}, m'_c = \frac{3c}{4}$. Then the inequality we have to prove is

$$({m_a}^2+{m_b}^2+{m_c}^2)\left(\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{1}{m_a m_b}\right)\le\frac{m'_a m'_b m'_c(m'_a+m'_b+m'_c)}{S'^2}.$$

Now, the last inequality will be true if the triangle with side − lengths a, b, c and area S satisfies the following inequality :

$$(a^2+b^2+c^2)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)\le\frac{m_a m_b m_c(m_a+m_b+m_c)}{S^2}.$$

By Tereshin's inequality, we have

$$m_a+m_b+m_c\ge\frac{b^2+c^2}{4R}+\frac{c^2+a^2}{4R}+\frac{a^2+b^2}{4R}=\frac{a^2+b^2+c^2}{2R}.$$

Using the known inequality, $m_a\ge\sqrt{s(s-a)}$ (and analogs), we have

$$m_a m_b m_c\ge\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}=\frac{S^2}{r}.$$

Then

$$\frac{m_a m_b m_c(m_a+m_b+m_c)}{S^2}\ge\frac{a^2+b^2+c^2}{2Rr}=(a^2+b^2+c^2)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right),$$

which completes the proof. Equality holds iff ΔABC is equilateral.

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