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In ΔABC holds:

$$\sum_{cyc} \sin^4 A.\sin(2A) \leq \frac{27\sqrt{3}}{32}$$

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Since $sin\,A = \frac{a}{2R}$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$ (and analogs), where R is the circumradius of ΔABC, then we have

$$\sum_{cyc} \sin^4 A.\sin(2A) = \sum_{cyc} \sin^5 A.2\cos A = \sum_{cyc} \frac{a^5(b^2+c^2-a^2)}{32R^5.bc} =$$

$$= \frac{1}{32R^5}\sum_{cyc} \frac{a^3[b^2c^2-(a^2-b^2)(a^2-c^2)]}{bc}$$

$$= \frac{abc(a^2+b^2+c^2)}{32R^5} - \frac{\sum_{cyc} a^4(a^2-b^2)(a^2-c^2)}{32R^5.abc}.$$

By Schur's inequality, we have

$$\sum_{cyc} a^4(a^2-b^2)(a^2-c^2) \geq 0.$$

By Leibniz's inequality, we have $a^2+b^2+c^2 \leq 9R^2$, and by Mitrinovic and Euler inequalities, we have $abc = R.2s.2r \leq R.3\sqrt{3}R.R = 3\sqrt{3}R^3$.

Using these results, we have

$$\sum_{cyc} \sin^4 A.\sin(2A) \leq \frac{abc(a^2+b^2+c^2)}{32R^5} \leq \frac{3\sqrt{3}R^3.9R^2}{32R^5} = \frac{27\sqrt{3}}{32},$$

as desired. Equality holds iff ΔABC is equilateral.