ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$108r^4 \le a^2(s-a)^2 + b^2(s-b)^2 + c^2(s-c)^2 \le \frac{27R^4}{4}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\sum a^2(s-a)^2 = s^2\sum a^2 - 2s\sum a^3 + \sum a^4 =$$

$$= 2s^2(s^2 - r^2 - 4Rr) - 4s^2(s^2 - 3r^2 - 6Rr) +$$
$$+2(s^4 - 6r^2s^2 - 8s^2Rr + 8Rr^3 + 16R^2r^2 + r^4) =$$

$$= 32R^2r^2 + 16Rr^3 + 2r^4 - 2r^2s^2 \quad (1)$$

$$\sum a^2(s-a)^2 = 32R^2r^2 + 16Rr^3 + 2r^4 - 2r^2s^2 \overset{Gerretsen}{\le}$$
$$\le 32R^2r^2 - 16Rr^3 + 12r^4$$

$$Now\ \ we\ need\ to\ show:$$
$$32R^2r^2 - 16Rr^3 + 12r^4 \le \frac{27R^4}{4}\ or$$

$$27x^4 - 128x^2 + 64x - 48 \overset{\frac{R}{r}=x\ge 2\ (Euler)}{\ge} 0\ or$$

$$(x-2)[27x^3 + x(54x-20) + 24] \ge 0\ True$$

$$\sum a^2(s-a)^2 = 32R^2r^2 + 16Rr^3 + 2r^4 - 2r^2s^2 \overset{Gerretsen}{\ge}$$

$$\ge 24R^2r^2 + 8Rr^3 - 4r^4 \overset{Euler}{\ge} 96r^4 + 16r^4 - 4r^4 = 108r^4$$

Equality holds for $a = b = c$.