ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$:27r^2 \le h_a^2 + h_b^2 + h_c^2 \le \frac{27R(R-r)}{2}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$h_a^2 + h_b^2 + h_c^2 = 4F^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \overset{Steining}{\le} \frac{4F^2}{4r^2} = \frac{r^2s^2}{r^2} = s^2 \overset{Mitrinovic}{\le}$$

$$\le \frac{27}{4}R^2 = \frac{27R}{2}\left(\frac{R}{2}\right) = \frac{27R}{2}\left(R - \frac{R}{2}\right) \overset{Euler}{\le} \frac{27R(R-r)}{2}$$

$$h_a + h_b + h_c \overset{AM-HM}{\ge} \frac{9}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} = \frac{9}{\frac{1}{r}} = 9r\ (1)$$

$$h_a^2 + h_b^2 + h_c^2 \overset{CBS}{\ge} \ge \frac{1}{3}(h_a + h_b + h_c)^2 \overset{(1)}{\ge} \frac{1}{3}(9r)^2 = 27r^2$$

$$Equality\ holds\ for\ a = b = c.$$