ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$h_a + h_b + h_c \leq 4R + r$$

Proposed by George Apostolopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$h_a + h_b + h_c = \frac{2S}{a} + \frac{2S}{b} + \frac{2S}{c}$$

On the other hand according to the known formulas for the triangle:

$$S = \frac{ar_br_c}{r_b + r_c} = \frac{br_ar_c}{r_a + r_c} = \frac{cr_ar_b}{r_a + r_b}$$

$$\frac{2S}{a} + \frac{2S}{b} + \frac{2S}{c} = \frac{2r_br_c}{r_b + r_c} + \frac{2r_ar_c}{r_a + r_c} + \frac{2r_ar_b}{r_a + r_b} \overset{A-G}{\leq} \frac{{r_b}^2 + {r_c}^2}{r_b + r_c} + \frac{{r_a}^2 + {r_c}^2}{r_a + r_c} +$$

$$+\frac{{r_a}^2 + {r_b}^2}{r_a + r_b} = \left(r_b + r_c - \frac{2r_br_c}{r_b + r_c}\right) + \left(r_a + r_c - \frac{2r_ar_c}{r_a + r_c}\right) + \left(r_a + r_b - \frac{2r_ar_b}{r_a + r_b}\right) =$$

$$= 2(r_a + r_b + r_c) - \left(\frac{2r_br_c}{r_b + r_c} + \frac{2r_ar_c}{r_a + r_c} + \frac{2r_ar_b}{r_a + r_b}\right) =$$

$$= 2(r_a + r_b + r_c) - (h_a + h_b + h_c) \Longrightarrow 2(h_a + h_b + h_c) \leq 2(r_a + r_b + r_c)$$

$$h_a + h_b + h_c \leq r_a + r_b + r_c = 4R + r \quad (proved)$$

Equality holds for $a = b = c$.