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In ΔABC the following relationship holds:

$$\cos(B-C)\cos(C-A)\cos(A-B)\geq 8\cos A\cos B\cos C$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\frac{b\cos B+c\cos C}{a}=\frac{R(\sin 2B+\sin 2C)}{2R\sin A}=$$

$$=R\frac{2\sin(B+C)\cos(B-C)}{2R\sin A}\overset{A+B+C=\pi}{=}\cos(B-C)\quad(1)$$

$$\cos(B-C)\cos(C-A)\cos(A-B)=\prod\cos(B-C)\overset{(1)}{=}$$

$$=\prod\frac{b\cos B+c\cos C}{a}\overset{AM-GM}{\geq}\prod\frac{2\sqrt{bc\cos A\cos C}}{a}=8\cos A\cos B\cos C$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$