ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{9}{4}\left(\frac{R}{2r}\right)^{-2}-\frac{15}{4}\le \cos 2A+\cos 2B+\cos 2C\le 5-\frac{13}{2}\left(\frac{R}{2r}\right)^{-2}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\begin{aligned}\cos 2A+\cos 2B+\cos 2C&=2\cos(A+B)\cos(A-B)+\cos 2C\\&=2\cos(\pi-C)\cos(A-B)+2\cos^2 C-1\\&=-2\cos C\,(\cos(A-B)-\cos C)-1\\&=-2\cos C\,(\cos(A-B)+\cos(A+B))-1\\&=-4\cos A\cos B\cos C-1\quad(1)\end{aligned}$$

$$From\ (1):\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1=$$

$$=4\frac{(2R+r)^2-s^2}{4R^2}-1\overset{Gerretsen}{\le}\frac{4R^2+4Rr+r^2-16Rr+5r^2}{R^2}-1=$$

$$=\frac{3R^2-12Rr+6r^2}{R^2}=3-12\frac{r}{R}+6\left(\frac{r}{R}\right)^2=5-2-12\frac{r}{R}+6\left(\frac{r}{R}\right)^2=$$

$$=5-8.\left(\frac{1}{2}\right)^2-24\left(\frac{1}{2}\right).\left(\frac{r}{R}\right)+6\left(\frac{r}{R}\right)^2\overset{Euler}{\le}$$

$$\le 5-8\left(\frac{r}{R}\right)^2-24\left(\frac{r}{R}\right)^2+6\left(\frac{r}{R}\right)^2=5-26\left(\frac{r}{R}\right)^2=5-\frac{13}{2}.4\left(\frac{r}{R}\right)^2==5-\frac{13}{2}\left(\frac{R}{2r}\right)^{-2}$$

$$From\ (1):co\,s\,2A+co\,s\,2B+co\,s\,2C=-4co\,s\,A\,co\,s\,B\,co\,s\,C-1=$$

$$=4\frac{(2R+r)^2-s^2}{4R^2}-1\overset{Gerretsen}{\ge}\frac{(2R+r)^2-4R^2-4Rr-3r^2}{R^2}-1=$$

$$=\frac{-2r^2-R^2}{R^2}=-2\left(\frac{r}{R}\right)^2-1=-2\left(\frac{r}{R}\right)^2-\left(\frac{15}{4}-\frac{11}{4}\right)=$$

$$=\frac{11}{4}-2\left(\frac{r}{R}\right)^2-\frac{15}{4}=11.\left(\frac{1}{2}\right)^2-2\left(\frac{r}{R}\right)^2-\frac{15}{4}\overset{Euler}{\ge}$$

$$\ge 11\cdot\left(\frac{r}{R}\right)^2-2\left(\frac{r}{R}\right)^2-\frac{15}{4}=9\left(\frac{r}{R}\right)^2-\frac{15}{4}=\frac{9}{4}\left(\frac{2r}{R}\right)^2-\frac{15}{4}=\frac{9}{4}\left(\frac{R}{2r}\right)^{-2}-\frac{15}{4}$$

Equality holds for: $a=b=c$.