ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$h_a(b+c)^2+h_b(c+a)^2+h_c(a+b)^2\le 54R^3$$

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Solution by Soumava Chakraborty-Kolkata-India

$$h_a(b+c)^2+h_b(c+a)^2+h_c(a+b)^2=\sum_{cyc}\frac{bc(b^2+c^2+2bc)}{2R}$$

$$\overset{A-G}{\le}\frac{1}{2R}\left(\sum_{cyc}\frac{(b^2+c^2)^2}{2}+2\sum_{cyc}a^2b^2\right)=\frac{1}{2R}\left(\sum_{cyc}a^4+2\sum_{cyc}a^2b^2+\sum_{cyc}a^2b^2\right)$$

$$=\frac{1}{2R}\left(\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2\right)\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Goldstone}}}{\le}\frac{1}{2R}(81R^4+4R^2s^2)$$

$$\overset{\text{Mitrinovic}}{\le}\frac{1}{2R}\left(81R^4+4R^2.\frac{27R^2}{4}\right)\therefore h_a(b+c)^2+h_b(c+a)^2+h_c(a+b)^2\le 54R^3$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)