ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)+h_b^2\left(\frac{1}{m_c^2}+\frac{1}{m_a^2}\right)+h_c^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)\geq 6$$

Proposed by Lam Tran-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} b^2c^2\sec^4\frac{A}{2}=\left(\sum_{cyc} bc\sec^2\frac{A}{2}\right)^2-2\sum_{cyc} bc.ca.\sec^2\frac{A}{2}\sec^2\frac{B}{2}$$

$$=\left(\sum_{cyc} bc+4Rrs.\sum_{cyc}\frac{\tan^2\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\right)^2-8Rrs.\frac{16R^2}{s^2}.\sum_{cyc} a\cos^2\frac{A}{2}$$

$$=\left(s^2+4Rr+r^2+r\sum_{cyc}\left(\left(1+\tan^2\frac{A}{2}\right)r_a\right)\right)^2-\frac{64R^3r}{s}\sum_{cyc} a(1+\cos A)$$

$$=\left(s^2+4Rr+r^2+r(4R+r)+\frac{r}{s^2}\left((4R+r)^3-3.64R^3.\frac{s^2}{16R^2}\right)\right)^2$$

$$-\frac{64R^3r}{s}\left(2s+\frac{2rs}{R}\right)\left(\because r_b+r_c=4R\cos^2\frac{A}{2}\text{ and analogs}\right)$$

$$\Rightarrow\sum_{cyc} b^2c^2\sec^4\frac{A}{2}=$$

$$\frac{\left(s^2(s^2+4Rr+r^2)+r(4R+r)s^2+r(4R+r)^3-12Rrs^2\right)^2-128R^2r(R+r)s^4}{s^4}$$

$$\rightarrow(1)$$

$$\text{Also,}\sum_{cyc}\sec^4\frac{A}{2}=\left(\sum_{cyc}\sec^2\frac{A}{2}\right)^2-2\sum_{cyc}\sec^2\frac{A}{2}\sec^2\frac{B}{2}$$

$$=\frac{\left(s^2+(4R+r)^2\right)^2}{s^4}-2.\frac{16R^2}{s^2}.\frac{4R+r}{2R}$$

$$\Rightarrow\sum_{cyc}\sec^4\frac{A}{2}=\frac{\left(s^2+(4R+r)^2\right)^2-16R(4R+r)s^2}{s^4}\rightarrow(2)$$

$$\text{Now,}\because m_a\leq 2R\cos^2\frac{A}{2}\text{ and analogs }\forall\text{ acute }\Delta\text{ ABC},\therefore\sum_{cyc} h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)$$

$$\geq\sum_{cyc}\left(\frac{b^2c^2}{4R^2}\left(\frac{1}{4R^2}\left(\sum_{cyc}\sec^4\frac{A}{2}-\sec^4\frac{A}{2}\right)\right)\right)$$

$$=\frac{\left(\sum_{cyc}\sec^4\frac{A}{2}\right)\left(\sum_{cyc}b^2c^2\right)-\sum_{cyc}b^2c^2\sec^4\frac{A}{2}}{16R^4}$$

$$\overset{via\ (1)\ and\ (2)}{=}\frac{\left(\left(s^2+(4R+r)^2\right)^2-16R(4R+r)s^2\right)\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)}{16R^4s^4}$$

$$-\frac{\left(s^2\left(s^2+4Rr+r^2\right)+r(4R+r)s^2+r(4R+r)^3-12Rrs^2\right)^2-128R^2r(R+r)s^4}{16R^4s^4}$$

$$=\frac{32R^2s^2\left(-s^4+\left(8R^2+16Rr+2r^2\right)s^2-r(4R+r)^3\right)}{16R^4s^4}\overset{?}{\geq}6$$

$$\Leftrightarrow \boxed{s^4-\left(5R^2+16Rr+2r^2\right)s^2+r(4R+r)^3\underset{(*)}{\overset{?}{\leq}}0}$$

Now, via Wallker and Gerretsen, $\forall$ acute Δ ABC,

$\left(s^2-2R^2-8Rr-3r^2\right)\left(s^2-4R^2-4Rr-3r^2\right)\leq 0$ and so,

in order to prove $(*)$, it suffices to prove :

LHS of $(*)\leq\left(s^2-2R^2-8Rr-3r^2\right)\left(s^2-4R^2-4Rr-3r^2\right)$

$$\Leftrightarrow\boxed{\left(R^2-4Rr+4r^2\right)s^2\overset{(**)}{\leq}8R^4-24R^3r+2R^2r^2+24Rr^3+8r^4}$$

Again, LHS of $(**)\overset{Gerretsen}{\leq}\left(R^2-4Rr+4r^2\right)\left(4R^2+4Rr+3r^2\right)\overset{?}{\leq}$

$8R^4-24R^3r+2R^2r^2+24Rr^3+8r^4\Leftrightarrow 4t^4-12t^3-t^2+20t-4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$

$\Leftrightarrow(t-2)^2\left(4t^2+4t-1\right)\overset{?}{\geq}0\rightarrow$ true $\because t\overset{Euler}{\geq}2\Rightarrow(**)\Rightarrow(*)$ is true

$$\therefore h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)+h_b^2\left(\frac{1}{m_c^2}+\frac{1}{m_a^2}\right)+h_c^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)\geq 6$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

We will first prove that, for any acute triangle ABC, we have

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2}$$

Since $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{(b+c)^2-a^2}{4bc}}$, then we have

$$\left(\sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2}\right)^2-{m_a}^2=\frac{(b^2+c^2)[(b+c)^2-a^2]}{8bc}-\frac{2\left(b^2+c^2\right)-a^2}{4}$$

$$=\frac{\left(b^2+c^2-a^2\right)(b-c)^2}{8bc}\geq 0,$$

which is true for the acute triangle ABC.

Using this result, and since $h_a=\frac{bc}{2R}$ (and analogs) and $a=4R\sin\frac{A}{2}\cos\frac{A}{2}$, we have

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2}=\sqrt{\frac{b^2+c^2}{2}}.\frac{a}{4R\sin\frac{A}{2}}=\sqrt{\frac{{h_c}^2+{h_b}^2}{8}}\csc\frac{A}{2}\quad\text{(and analogs)}$$

Therefore

$$\sum_{cyc} h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)=\sum_{cyc}\frac{h_b^2+h_c^2}{m_a^2}\ge\sum_{cyc}8\sin^2\frac{A}{2}=4\sum_{cyc}(1-\cos A)\overset{Jensen}{\ge}4\left(3-\frac{3}{2}\right)=6.$$

Equality holds iff ΔABC is equilateral.