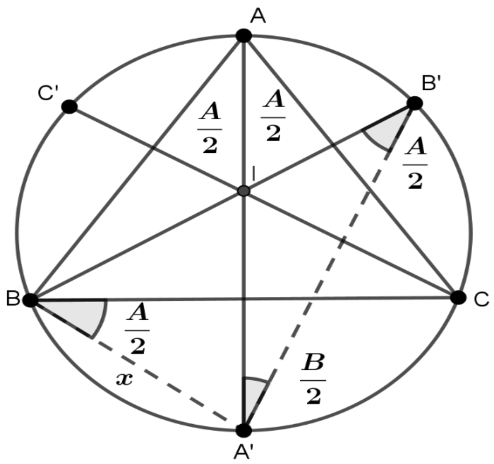
ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC, A', B', C'$ - middle points of the arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ respectively made with the circumcircle of the triangle ABC the following relationship holds:

$$\frac{6r}{R} \le \frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} \le 3$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Tapas Das – India



A', B', C' are the mid points of arc BC, CA, AB so, AA', BB', CC' are the angle of bisector

From $\Delta ABA'$ we have

$$\frac{x}{\sin\frac{A}{2}} = \frac{c}{\sin C} = 2R \Rightarrow x = 2R\sin\frac{A}{2}$$

Now

$$\angle BIA' = \pi - (\angle A'BI + \angle BA'I) = \pi - \left(\frac{A}{2} + \frac{B}{2}\right) - C = \pi - \frac{A}{2} - \frac{B}{2} - [\pi - (B + A)] = \frac{A+B}{2}$$

$$\therefore \angle A'BI = \angle AIB = \frac{A+B}{2}$$

$$\therefore A'B = A'I = 2R\sin\frac{A}{2} \quad \text{(analog)}$$

From $\Delta A'IB', \angle AA'B' = \frac{B}{2}, \angle BB'A' = \frac{A}{2}$

$$\therefore \angle A'IB' = \pi - \left(\frac{A+B}{2}\right)$$

$$\therefore \sin\angle A'IB' = \sin\frac{A+B}{2} = \cos\frac{C}{2}$$

$$(\because A+B+C=\pi)$$

From $\Delta A'IB'$

$$\frac{A'B'}{\sin\angle A'IB'} = \frac{IB'}{\sin\angle IA'B'} \Rightarrow \frac{A'B'}{\cos\frac{C}{2}} = \frac{2R\sin\frac{B}{2}}{\sin\frac{B}{2}}$$

$$\therefore A'B' = 2R\cos\frac{C}{2}$$

Now

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} = 2\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \overset{Jensen}{\leq} 2\times 3\cdot\sin\left(\frac{A+B+C}{6}\right) =$$

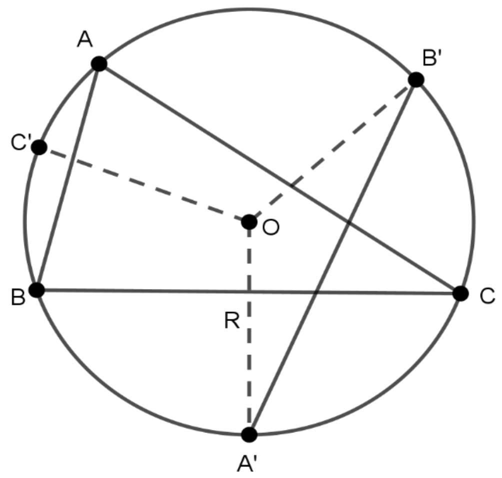
$$= 6\cdot\sin\frac{\pi}{6} = 6\times\frac{1}{2} = 3$$

Note $f(x)=\sin x$ is concave in $\left(0,\frac{\pi}{2}\right)$

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} = 2\left(\sum\sin\frac{A}{2}\right) \overset{AM-GM}{\geq} 6\left(\prod\sin\frac{A}{2}\right)^{\frac{1}{3}} = 6\times\left(\frac{r}{4R}\right)^{\frac{1}{3}} =$$

$$= 6\times\left(\frac{r^3}{4Rr^2}\right)^{\frac{1}{3}} \overset{Euler}{\geq} 6\times\left(\frac{r^3}{4R\cdot\frac{R^2}{4}}\right) = \frac{6r}{R}$$

Solution 2 by Adrian Popa – Romania



$$\widehat{A'OB} = \widehat{A'B'} = \widehat{A'C} = \widehat{CB'} = \frac{\widehat{BC}}{2} + \frac{\widehat{AC}}{2} = \widehat{A} + \widehat{B} = \pi - \widehat{C}$$

$\Delta OA'B'$ (Cosine Theorem)

$$A'B'^2 = OA'^2 + OB'^2 - 2OA' \cdot OB' \cdot \cos \widehat{A'OB} \Rightarrow$$

$$\Rightarrow A'B'^2 = R^2 + R^2 + 2R \cdot R \cos C - 2R^2(1 + \cos C) =$$

$$= 2R^2 \cdot 2\cos^2\frac{C}{2} = 4R^2\cos^2\frac{C}{2} \Rightarrow A'B' = 2R\cos\frac{C}{2}$$

Similarly: $A'C' = 2R\cos\frac{B}{2}$ and $B'C' = 2R\cos\frac{A}{2}$

$$\frac{AB}{A'B'} + \frac{BC}{B'C'} + \frac{CA}{C'A'} = \sum_{cyc}\frac{2R\sin C}{2R\cos\frac{C}{2}} = \sum_{cyc}\frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{C}{2}} = 2\sum_{cyc}\sin\frac{C}{2} = 2\sum_{cyc}\sin\frac{A}{2}$$

We must show that: $\frac{6r}{R} \overset{(1)}{\le} 2\sum\sin\frac{A}{2} \overset{(2)}{\le} 3$

$$(1)\quad \sum\sin\frac{A}{2} \overset{MA\ge MG}{\ge} 3\sqrt[3]{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = 3\sqrt[3]{\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)(s-c)}{ac}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}}$$

$$= 3\sqrt[3]{\frac{S^2}{4RS\cdot s}} = 3\sqrt[3]{\frac{r}{4R}} = 3\sqrt[3]{\frac{r^3}{4Rr^2}} = \frac{3r}{\sqrt[3]{4Rr^2}} \overset{R\ge 2r}{\ge} \frac{3r}{\sqrt[3]{4R\frac{R^2}{4}}} = \frac{3r}{R}$$

$$\Rightarrow \sum\sin\frac{A}{2} \ge \frac{3r}{R} \,|\cdot 2 \Rightarrow 2\sum\sin\frac{A}{2} \ge \frac{6r}{R}$$

$$(2)\ \sum\sin\frac{A}{2} \overset{Jensen}{\le} 3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3} = 3\sin\frac{A+B+C}{6} = 3\sin\frac{\pi}{6} = \frac{3}{2} \Rightarrow 2\sum\sin\frac{A}{2} \le 3$$