ROMANIAN MATHEMATICAL MAGAZINE

If in ΔABC: $a=\frac{b+c}{2}$ then:

$$\sin^2\frac{A}{2}\ge\frac{r}{2R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Tapas Das-India

Lemma:

$$\cos\frac{B-C}{2}\ge\sqrt{\frac{2r}{R}}$$

$(Reference:\ Geometric\ Inequalities\ Marathon, First\ 100\ problems\ and\ solutions)(1)$

$$Mollweide's:\quad\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\frac{b+c}{a}$$

$$Now\ a=\frac{b+c}{2}\ or\quad\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\frac{b+c}{a}=2$$

$$or\ \ 2\sin\frac{A}{2}=\cos\frac{B-C}{2}\overset{(1)}{\ge}\sqrt{\frac{2r}{R}}\ or\sin^2\frac{A}{2}\ge\frac{r}{2R}$$

Solution 2 by Tapas Das-India

$$\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}=$$

$$=\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}.+\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}=$$

$$=\frac{1}{a\sqrt{bc}}(2s-a)\sqrt{(s-b)(s-c)}=\frac{\frac{1}{a\sqrt{bc}}(b+c)\left(\sqrt{s(s-a)(s-b)(s-c)}\right)}{\sqrt{s(s-a)}}=$$

$$=\frac{\frac{1}{a\sqrt{bc}}(b+c)2\left(\sqrt{s(s-a)(s-b)(s-c)}\right)}{2\sqrt{s(s-a)}}=\frac{\frac{2F}{a}(b+c)}{2\sqrt{bcs(s-a)}}.=\frac{h_a}{w_a}=$$

$$=\frac{(b+c)bc}{2R.2\sqrt{bcs(s-a)}.}=\frac{(2(s-a)+a)\sqrt{bc}}{4R\sqrt{s(s-a)}}\overset{AM-GM}{\ge}$$

$$\geq \frac{2\sqrt{2(s-a)abc}}{4R\sqrt{s(s-a)}} = \frac{1}{2R}\sqrt{\frac{8Rsr}{s}} = \sqrt{\frac{2r}{R}}\ (1)$$

$$Mollweide's: \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}} = \frac{b+c}{a}$$

$$Now\ a = \frac{b+c}{2}\ or\ \ \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}} = \frac{b+c}{a} = 2$$

$$or\ \ 2\sin\frac{A}{2} = \cos\frac{B-C}{2} \overset{(1)}{\geq} \sqrt{\frac{2r}{R}}\ or \sin^2\frac{A}{2} \geq \frac{r}{2R}$$