ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, p_b, p_c → Spieker cevians, the following relationship holds :

$$\frac{p_a - m_a + w_a}{h_a} + \frac{p_b - m_b + w_b}{h_b} + \frac{p_c - m_c + w_c}{h_c} \le \frac{R}{r} + 1$$

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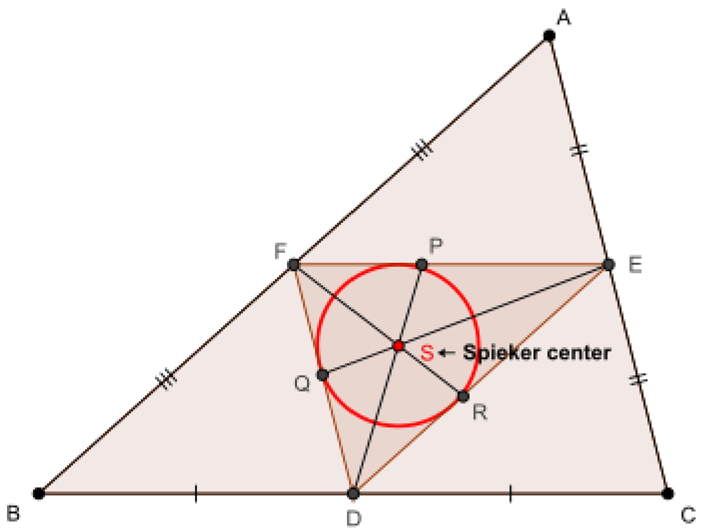

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS,}\frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Also, } p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right) - \left(1 - \frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)\left(b^3+c^3-abc\right) - \left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right) + \left((b+c)^2+2a(b+c)+a^2\right) - a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2 - a^2\right) = \frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\blacksquare\blacksquare)}{=} \frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a$$

$$\therefore \text{ in order to prove : } \frac{p_a^2 - m_a^2}{p_a + m_a} \leq \frac{m_a^2 - w_a^2}{m_a + w_a}, \text{ it suffices to prove : }$$

$$p_a^2 - m_a^2 \leq m_a^2 - w_a^2$$

$$\overset{\text{via } (\blacksquare\blacksquare)}{\Leftrightarrow} \frac{(b-c)^2\left(8s^2-a^2\right)}{4(2s+a)^2} \leq s(s-a) + \frac{(b-c)^2}{4} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$= \frac{(b-c)^2}{4}\left(1 + \frac{4s(s-a)}{(2s-a)^2}\right) = \frac{(b-c)^2}{4}\cdot\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow \left((2s-a)^2+4s(s-a)\right)(2s+a)^2 \geq \left(8s^2-a^2\right)(2s-a)^2$$

$$\Leftrightarrow 16s^3 - 12s^2a - 4sa^2 + a^3 \geq 0 \Leftrightarrow (s-a)\left(16s^2+4sa\right) + a^3 \geq 0$$

$$\rightarrow \text{ true (strict) since } s > a \therefore \frac{p_a^2 - m_a^2}{p_a + m_a} \leq \frac{m_a^2 - w_a^2}{m_a + w_a} \Rightarrow p_a + w_a \leq 2m_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\frac{p_a - m_a + w_a}{h_a} \leq \sum_{cyc}\frac{m_a}{h_a} = \frac{1}{2rs}\sum_{cyc}\left(\sqrt{a}m_a.\sqrt{a}\right) \overset{CBS}{\leq}$$

$$\frac{1}{4rs}\cdot\sqrt{\sum_{cyc}a(2b^2+2c^2-a^2)}\cdot\sqrt{\sum_{cyc}a}$$

$$= \frac{\sqrt{2s}}{4rs}\cdot\sqrt{2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) - 6abc - 2s(s^2-6Rr-3r^2)}$$

$$=\frac{\sqrt{2s}.\sqrt{2s}}{4rs}.\sqrt{2(s^2+4Rr+r^2)-12Rr-s^2+6Rr+3r^2}\overset{\text{Gerretsen}}{\leq}$$

$$\frac{1}{2r}.\sqrt{4R^2+4Rr+3r^2+2Rr+5r^2}=\frac{1}{2r}.\sqrt{4R^2+6Rr+4r^2+4r^2}$$

$$\overset{\text{Euler}}{\leq}\frac{1}{2r}.\sqrt{4R^2+6Rr+4r^2+2Rr}=\frac{1}{2r}.\sqrt{4R^2+8Rr+4r^2}=\frac{1}{r}.\sqrt{(R+r)^2}$$

$$=\frac{R}{r}+1\therefore\frac{p_a-m_a+w_a}{h_a}+\frac{p_b-m_b+w_b}{h_b}+\frac{p_c-m_c+w_c}{h_c}\leq\frac{R}{r}+1$$

$\forall\,\Delta\,ABC,''='' $ iff Δ ABC is equilateral (QED)