ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r_b+r_c}{2r_a+n_a}+\frac{r_c+r_a}{2r_b+n_b}+\frac{r_a+r_b}{2r_c+n_c}\geq 2$$

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$$\frac{r_b+r_c}{2r_a+n_a}+\frac{r_c+r_a}{2r_b+n_b}+\frac{r_a+r_b}{2r_c+n_c}=\sum_{cyc}\frac{(r_b+r_c)^2}{2r_a(r_b+r_c)+n_a(r_b+r_c)}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}(r_b+r_c)\right)^2}{2\sum_{cyc}r_a(r_b+r_c)+\sum_{cyc}n_a(r_b+r_c)}=\frac{4(4R+r)^2}{4s^2+\sum_{cyc}n_a(r_b+r_c)}\overset{?}{\geq}2$$

$$\Leftrightarrow \boxed{2(4R+r)^2-4s^2\underset{(*)}{\overset{?}{\geq}}\sum_{cyc}n_a(r_b+r_c)}$$

$$\text{Now, } r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Also, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$

$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$=as^2-2ah_ar_a\Rightarrow n_a^2=s^2-2h_ar_a=s^2-\frac{4rs^2\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=s^2\left(1-\frac{r}{R}\sec^2\frac{A}{2}\right)$$

$$\Rightarrow n_a(r_b+r_c)\overset{\text{via (i)}}{=}4Rs.\sqrt{1-\frac{r}{R}\sec^2\frac{A}{2}}.\cos^2\frac{A}{2}\ \text{and analogs}$$

$$\Rightarrow\sum_{cyc}n_a(r_b+r_c)=4Rs.\sum_{cyc}\left(\left(\sqrt{1-\frac{r}{R}\sec^2\frac{A}{2}}.\cos\frac{A}{2}\right)\left(\cos\frac{A}{2}\right)\right)\overset{\text{CBS}}{\leq}$$

$$4Rs.\sqrt{\sum_{cyc}\left(\left(1-\frac{r}{R}\sec^2\frac{A}{2}\right)\cos^2\frac{A}{2}\right)}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}$$

$$=4Rs.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}-\frac{3r}{R}}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}=4Rs.\sqrt{\left(\frac{4R+r}{2R}-\frac{3r}{R}\right)\left(\frac{4R+r}{2R}\right)}$$

$$=2s.\sqrt{(4R+r)(4R-5r)}\overset{?}{\leq}2(4R+r)^2-4s^2$$

$$\Leftrightarrow (4R+r)^4+4s^4-4s^2(4R+r)^2 \overset{?}{\geq} s^2(4R+r)(4R-5r)$$

$$\Leftrightarrow \boxed{4s^4+(4R+r)^4-s^2(4R+r)(20R-r) \underset{(**)}{\overset{?}{\geq}} 0} \; and$$

$$\because 4\left(s^2-4R^2-4Rr-3r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{ in order to prove } (**), \text{it suffices to prove :}$$

$$4s^4+(4R+r)^4-s^2(4R+r)(20R-r) \geq 4\left(s^2-4R^2-4Rr-3r^2\right)^2$$

$$\Leftrightarrow \boxed{\left(48R^2-16Rr-25r^2\right)s^2 \overset{(***)}{\leq} 192R^4+128R^3r-64R^2r^2-80Rr^3-35r^4}$$

$$\text{Now, LHS of } (***) \overset{\text{Rouche}}{\leq} \left(48R^2-16Rr-25r^2\right)\begin{pmatrix} 2R^2+10Rr-r^2 \\ +2(R-2r).\sqrt{R^2-2Rr} \end{pmatrix}$$

$$\overset{?}{\leq} 192R^4+128R^3r-64R^2r^2-80Rr^3-35r^4$$

$$\Leftrightarrow \boxed{(R-2r)\left(48R^3-64R^2r-31Rr^2+15r^3\right) \underset{(****)}{\overset{?}{\geq}} (R-2r)\left(48R^2-16Rr-25r^2\right).\sqrt{R^2-2Rr}}$$

$$\because 48R^3-64R^2r-31Rr^2+15r^3 = (R-2r)\left(48R^2+32Rr+33r^2\right)+81r^3$$

$$\overset{\text{Euler}}{\geq} 81r^3 > 0 \; and \because R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{ in order to prove } (****), \text{it suffices to prove :}$$

$$\left(48R^3-64R^2r-31Rr^2+15r^3\right)^2 > \left(R^2-2Rr\right)\left(48R^2-16Rr-25r^2\right)^2$$

$$\Leftrightarrow r^2\left(192R^4+320R^3r+16R^2r^2+320Rr^3+225r^4\right) > 0 \rightarrow \text{true}$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{is true}$$

$$\therefore \frac{r_b+r_c}{2r_a+n_a}+\frac{r_c+r_a}{2r_b+n_b}+\frac{r_a+r_b}{2r_c+n_c} \geq 2$$

$\forall \Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)