ROMANIAN MATHEMATICAL MAGAZINE

In any non − obtuse Δ ABC, the following relationship holds :

$$h_a^2 + h_b^2 + h_c^2 + \frac{18F^2}{m_a^2 + m_b^2 + m_c^2} \ge 8F + 4R^2 \cos A \cos B \cos C$$

When does equality hold ?

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$\boxed{\text{Case 1}}$ Δ ABC is right triangle and WLOG we may assume A = 90°; then :

$$a^2 = b^2 + c^2 \overset{A-G}{\ge} \frac{2bca}{a} \Rightarrow 8R^3 \sin^3 90° = 8Rrs \Rightarrow R^2 \ge r(2R + r)$$

$$\Rightarrow t^2 - 2t - 1 \ge 0 \text{ with equality iff } \Delta\, ABC \text{ is right isosceles} \to (i)$$

$$\Delta\, ABC \text{ is right triangle} \Rightarrow s = 2R + r \text{ and}$$

$$h_a^2 + h_b^2 + h_c^2 + \frac{18F^2}{m_a^2 + m_b^2 + m_c^2} \ge 8F + 4R^2 \cos A \cos B \cos C$$

$$\Leftrightarrow \frac{\left(s^2 + 4Rr + r^2\right)^2 - 16Rrs^2}{4R^2} + \frac{18r^2s^2}{\frac{3}{2}(s^2 - 4Rr - r^2)} \ge 8rs$$

$$\Leftrightarrow \frac{\left((2R + r)^2 + 4Rr + r^2\right)^2 - 16Rr(2R + r)^2}{4R^2} + \frac{12r^2(2R + r)^2}{4R^2} \ge 8r(2R + r)$$

$$\Leftrightarrow R^4 - 4R^3r + 2R^2r^2 + 4Rr^3 + r^4 \ge 0 \Leftrightarrow \boxed{\left(t^2 - 2t - 1\right)^2 \ge 0} \left(t = \frac{R}{r}\right) \to \text{true}$$

with equality iff Δ ABC is right isosceles (via (i))

$\boxed{\text{Case 2}}$ Δ ABC is acute ⇒ $b^2 + c^2 > a^2$ and analogs ∴ a^2, b^2, c^2 form sides of a triangle XYZ (say) ∴ $h_a^2 + h_b^2 + h_c^2 + \frac{18F^2}{m_a^2 + m_b^2 + m_c^2} - 4R^2 \cos A \cos B \cos C$

$$= \frac{\sum_{cyc} a^2b^2}{\frac{4a^2b^2c^2}{2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4}} + \frac{\frac{18}{16}\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)}{\frac{3}{4}\sum_{cyc} a^2} - \left(\left(s^2 - 4Rr - r^2\right) - 4R^2\right)$$

$$= \frac{\left(\sum_{yc} xy\right)\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{4xyz} + \frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{\sum_{cyc} x}$$

$$-\frac{\sum_{cyc} a^2}{2} + \frac{4xyz}{2\sum_{yc} xy - \sum_{cyc} x^2}$$

$$= \frac{\left(\sum_{yc} xy\right)\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{4xyz} + \frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{\sum_{cyc} x} - \frac{\sum_{cyc} x}{2}$$

$$+ \frac{4xyz}{2\sum_{yc} xy - \sum_{cyc} x^2} \ge 8F = 2.\sqrt{2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4} = 2.\sqrt{2\sum_{yc} xy - \sum_{cyc} x^2}$$

$$\Leftrightarrow \frac{\left(\sum_{yc} xy\right)\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{4xyz} + \frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{\sum_{cyc} x} - \frac{\sum_{cyc} x}{2}$$

$$+\frac{4xyz}{2\sum_{yc}xy-\sum_{cyc}x^2}\overset{(*)}{\geq}2.\sqrt{2\sum_{yc}xy-\sum_{cyc}x^2}$$

Now, we shall prove that $\forall\,\Delta\,ABC$:

$$\frac{\left(\sum_{yc}ab\right)\left(2\sum_{yc}ab-\sum_{cyc}a^2\right)}{4abc}+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc}ab-\sum_{cyc}a^2\right)}{\sum_{cyc}a}-\frac{\sum_{cyc}a}{2}$$

$$+\frac{4abc}{2\sum_{yc}ab-\sum_{cyc}a^2}\overset{(\bullet)}{\geq}2.\sqrt{2\sum_{yc}ab-\sum_{cyc}a^2}$$

$$(\bullet)\Leftrightarrow\frac{(s^2+4Rr+r^2)\left(2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)\right)}{16Rrs}$$

$$+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot(16Rr+4r^2)}{2s}-s+\frac{16Rrs}{16Rr+4r^2}\geq 2.\sqrt{16Rr+4r^2}$$

$$\Leftrightarrow\frac{(s^2+16Rr+r^2)(4R+r)^2-4R(4R+r)s^2+16R^2s^2}{4Rs(4R+r)}\geq 2.\sqrt{16Rr+4r^2}$$

$$\Leftrightarrow\frac{(4R+r)\left((4R+r)(16Rr+r^2)+rs^2\right)+16R^2s^2}{4Rs(4R+r)}\geq 2.\sqrt{16Rr+4r^2}\Leftrightarrow$$

$$\left((4R+r)\left((4R+r)(16Rr+r^2)+rs^2\right)+16R^2s^2\right)^2\geq 256(4Rr+r^2)R^2s^2(4R+r)^2$$

$$\Leftrightarrow(256R^4+128R^3r+48R^2r^2+8Rr^3+r^4)s^4$$

$$-rs^2(8192R^5+5632R^4r+640R^3r^2-256R^2r^3-56Rr^4-2r^5)$$

$$+r^2\begin{pmatrix}65536R^6+73728R^5r+33024R^4r^2+7424R^3r^3+864R^2r^4\\+48R^5r+r^6\end{pmatrix}\overset{(\bullet\bullet)}{\geq}0$$

$\because$ LHS of $(\bullet\bullet)$ is a quadratic polynomial in s^2 $\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : $r^2(8192R^5+5632R^4r+640R^3r^2-256R^2r^3-56Rr^4-2r^5)^2$

$$-4\begin{pmatrix}256R^4+128R^3r+\\48R^2r^2+8Rr^3+r^4\end{pmatrix}.r^2\begin{pmatrix}65536R^6+73728R^5r+33024R^4r^2\\+7424R^3r^3+864R^2r^4+48R^5r+r^6\end{pmatrix}<0$$

(i. e., discriminant <0)

$$\Leftrightarrow -1024R^2r^3\begin{pmatrix}16384R^7+40960R^6r+36864R^5r^2+\\16640R^4r^3+4160R^3r^4+576R^2r^5+40Rr^6+r^7\end{pmatrix}<0\rightarrow\text{true}$$

$\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\Rightarrow(*)$ is true (strict inequality)

$$\therefore h_a^2+h_b^2+h_c^2+\frac{18F^2}{m_a^2+m_b^2+m_c^2}>8F+4R^2\cos A\cos B\cos C\ \therefore$$

combining both cases, $h_a^2+h_b^2+h_c^2+\dfrac{18F^2}{m_a^2+m_b^2+m_c^2}\geq 8F+4R^2\cos A\cos B\cos C$

$\forall$ non $-$ obtuse Δ ABC," $=$" iff Δ ABC is right isosceles (QED)