# ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC, the following relationship holds :

$$1+\frac{4R}{r}\ge\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\ge\frac{5\sqrt{3}s}{6r}+\frac{3}{2}$$

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$$\sum_{cyc}\cot\frac{A}{2}=\sum_{cyc}\frac{s}{r_a}=\frac{s}{r}\Rightarrow\sum_{cyc}\cot\frac{A}{2}=\frac{s}{r}\to(1)\ and\ \sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}$$

$$=\sum_{cyc}\frac{s}{r_a}\cdot\frac{s}{r_b}=\frac{s^2}{rs^2}\cdot\sum_{cyc}r_a\Rightarrow\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}=\frac{4R+r}{r}\to(2)\ and\ also,$$

$$\prod_{cyc}\cot\frac{A}{2}=\prod_{cyc}\frac{s}{r_a}=\frac{s^3}{rs^2}\therefore\prod_{cyc}\cot\frac{A}{2}=\frac{s}{r}\to(3)$$

$$\text{Now, }\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}$$

$$=\sum_{cyc}\left(\sqrt{\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3\right)\cot\frac{A}{2}}\cdot\sqrt{\tan\frac{A}{2}}\right)$$

$$\overset{CBS}{\le}\sqrt{\sum_{cyc}\left(\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3\right)\cot\frac{A}{2}\right)}\cdot\sqrt{\sum_{cyc}\tan\frac{A}{2}}$$

$$=\sqrt{3\sum_{cyc}\cot\frac{A}{2}+\left(\sum_{cyc}\cot\frac{A}{2}\right)\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)-3\prod_{cyc}\cot\frac{A}{2}}\cdot\sqrt{\frac{4R+r}{s}}$$

$$\overset{\text{via }(1),(2)\ and\ (3)}{=}\sqrt{\frac{3s}{r}+\frac{s}{r}\cdot\frac{4R+r}{r}-\frac{3s}{r}}\cdot\sqrt{\frac{4R+r}{s}}$$

$$\therefore\boxed{\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\le\frac{4R+r}{r}}$$

$$\text{Again, }\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}$$

$$\ge\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\Leftrightarrow 2\sum_{cyc}\cot^2\frac{A}{2}+9+$$

$$2\sum_{cyc}\sqrt{\left(\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3\right)\left(\cot^2\frac{A}{2}+\cot^2\frac{C}{2}+3\right)}\overset{(*)}{\ge}\left(\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\right)^2$$

$$\text{Now, via Reverse CBS, } 2\sum_{cyc}\sqrt{\left(\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3\right)\left(\cot^2\frac{A}{2}+\cot^2\frac{C}{2}+3\right)}$$

$$\geq 2\sum_{cyc}\left(\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+3\right)\Rightarrow \text{LHS of } (*)$$

$$\geq 4\sum_{cyc}\cot^2\frac{A}{2}+27+2\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}$$

$$=\left(4\sum_{cyc}\cot^2\frac{A}{2}+8\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)-6\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}+27$$

$$=4\left(\sum_{cyc}\cot\frac{A}{2}\right)^2-6\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}+27\overset{\text{via (1) and (2)}}{=}\frac{4s^2}{r^2}-\frac{6(4R+r)}{r}+27$$

$$=\frac{4s^2-24Rr+21r^2}{r^2}\overset{?}{\geq}\left(\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\right)^2=\frac{25s^2+27r^2+30\sqrt{3}rs}{12r^2}$$

$$\Leftrightarrow \boxed{23s^2-288Rr+225r^2\underbrace{\overset{?}{\geq}}_{(**)}30\sqrt{3}rs} \text{ and } \because 23s^2-288Rr+225r^2\overset{\text{Gerretsen}}{\geq}$$

$$23(16Rr-5r^2)-288Rr+225r^2=80Rr+110r^2>0$$

$$\therefore (**)\Leftrightarrow (23s^2-288Rr+225r^2)^2\geq 2700r^2s^2$$

$$\Leftrightarrow 529s^4-(13248Rr-7650r^2)s^2+r^2(82944R^2-129600Rr+50625r^2)$$

$$\boxed{\overset{(***)}{\geq}}\ 0 \text{ and } \because 529(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore \text{ in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***)\geq 529(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (92R+59r)s^2\ \boxed{\overset{(****)}{\geq}}\ r(1312R^2+1124Rr-935r^2)$$

$$\text{Again, } (92R+59r)s^2\overset{\text{Gerretsen}}{\geq}(92R+59r)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(1312R^2+1124Rr-935r^2)\Leftrightarrow 6400r^2(R-2r)^2\overset{?}{\geq}0\to \text{true}$$

$$\Rightarrow (****)\Rightarrow (***)\Rightarrow (**)\Rightarrow (*) \text{ is true}$$

$$\therefore \boxed{\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}} \text{ and so,}$$

$$1+\frac{4R}{r}\geq\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}$$

$$\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\ \forall\,\Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$