ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a}{9R^2-a^2}+\frac{b}{9R^2-b^2}+\frac{c}{9R^2-c^2}\le\frac{9}{2\sqrt{3(a^2+b^2+c^2)}}$$

(Inspired by a problem of *Zaza Mzhavanadze*)

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$$\frac{a}{9R^2-a^2}+\frac{b}{9R^2-b^2}+\frac{c}{9R^2-c^2}=$$

$$=\frac{\sum_{cyc}\left(a\left(81R^4-9R^2(b^2+c^2)+b^2c^2\right)\right)}{729R^6-81R^4\sum_{cyc}a^2+9R^2\sum_{cyc}b^2c^2-16R^2r^2s^2}$$

$$=\frac{2s.81R^4-9R^2\left(2s(s^2+4Rr+r^2)-12Rrs\right)+4Rrs(s^2+4Rr+r^2)}{729R^6-162R^4(s^2-4Rr-r^2)+9R^2((s^2+4Rr+r^2)^2-16Rrs^2)-16R^2r^2s^2}$$

$$\le\frac{9}{2\sqrt{3(a^2+b^2+c^2)}}$$

$$\Leftrightarrow\left(\frac{2s.81R^4-9R^2\left(2s(s^2+4Rr+r^2)-12Rrs\right)+4Rrs(s^2+4Rr+r^2)}{729R^6-162R^4(s^2-4Rr-r^2)+9R^2((s^2+4Rr+r^2)^2-16Rrs^2)-16R^2r^2s^2}\right)^2$$

$$\le\frac{27}{8(s^2-4Rr-r^2)}$$

$$\Leftrightarrow 14348907R^{10}+25509168R^9r+23383404R^8r^2+13541904R^7r^3+$$
$$5401890R^6r^4+1504656R^5r^5+288684R^4r^6+34992R^3r^7+2187R^2r^8$$
$$-(405R^2-1152Rr+128r^2)s^8$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6$$
$$+\begin{pmatrix}852930R^6+664848R^5r+219348R^4r^2+29520R^3r^3-\\94R^2r^4-128Rr^5+2796r^6\end{pmatrix}s^4$$
$$-\begin{pmatrix}6377292R^8+7663248R^7r+4534380R^6r^2+1635552R^5r^3\\+283572R^4r^4+8656R^3r^5-2796R^2r^6-384Rr^7-128r^8\end{pmatrix}s^2\overset{(*)}{\ge}0$$

$$\text{Now, }-(405R^2-1152Rr+128r^2)s^8$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6$$
$$=-(405R-342r)(R-2r)s^8+556r^2s^8$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6\overset{\text{Gerretsen}}{\ge}$$
$$-(405R-342r)(R-2r)(4R^2+4Rr+3r^2)s^6+556r^2(16Rr-5r^2)s^6$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6$$
$$=-(33696R^4+21636R^3r+3267R^2r^2-10256Rr^3+4960r^4)s^6$$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$14348907R^{10}+25509168R^9r+23383404R^8r^2+13541904R^7r^3$$
$$+5401890R^6r^4+1504656R^5r^5+288684R^4r^6+34992R^3r^7+2187R^2r^8$$
$$-(33696R^4+21636R^3r+3267R^2r^2-10256Rr^3+4960r^4)s^6$$
$$+\begin{pmatrix}852930R^6+664848R^5r+219348R^4r^2+29520R^3r^3\\-94R^2r^4-128Rr^5+2796r^6\end{pmatrix}s^4$$
$$-\begin{pmatrix}6377292R^8+7663248R^7r+4534380R^6r^2+1635552R^5r^3+\\283572R^4r^4+8656R^3r^5-2796R^2r^6-384Rr^7-128r^8\end{pmatrix}s^2\overset{(**)}{\ge}0\ \text{and}$$

$$\because-\begin{pmatrix}33696R^4+21636R^3r+3267R^2r^2\\-10256Rr^3+4960r^4\end{pmatrix}(s^2-4R^2-4Rr-3r^2)^3$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \geq$$

$$-\begin{pmatrix} 33696R^4 + 21636R^3r + 3267R^2r^2 \\ -10256Rr^3 + 4960r^4 \end{pmatrix}(s^2 - 4R^2 - 4Rr - 3r^2)^3 \Leftrightarrow$$

$$12192363R^{10} + 17654832R^9r + 7698348R^8r^2 - 5559664R^7r^3 - 10151374R^6r^4$$
$$-6243072R^5r^5 - 1837528R^4r^6 - 63424R^3r^7 - 228294R^2r^8$$
$$-258768Rr^9 - 133920r^{10}$$
$$+\begin{pmatrix} 448578R^6 + 864R^5r - 382752R^4r^2 - 81336R^3r^3 + 34055R^2r^4 \\ +32656Rr^5 - 44512r^6 \end{pmatrix}s^4$$
$$-\begin{pmatrix} 4759884R^8 + 3389904R^7r - 1743012R^6r^2 - 3208224R^5r^3 \\ -1829556R^4r^4 - 56180R^3r^5 + 52227R^2r^6 - 80592Rr^7 - 134048r^8 \end{pmatrix}s^2 \overset{(***)}{\geq} 0 \text{ and}$$

$$\because 448578R^6 + 864R^5r - 382752R^4r^2 - 81336R^3r^3 +$$
$$34055R^2r^4 + 32656Rr^5 - 44512r^6$$
$$= (R - 2r)\begin{pmatrix} 448578R^5 + 898020R^4r + 1413288R^3r^2 + \\ 2745240R^2r^3 + 5524535Rr^4 + 11081726r^5 \end{pmatrix} + 22118940r^6$$
$$\overset{\text{Euler}}{\geq} 22118940r^6 > 0 \therefore$$
$$\begin{pmatrix} 448578R^6 + 864R^5r - 382752R^4r^2 - 81336R^3r^3 \\ +34055R^2r^4 + 32656Rr^5 - 44512r^6 \end{pmatrix}(s^2 - 4R^2 - 4Rr - 3r^2)^2$$
$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***), \text{it suffices to prove : LHS of } (***) \geq$$
$$\begin{pmatrix} 448578R^6 + 864R^5r - 382752R^4r^2 - 81336R^3r^3 + \\ 34055R^2r^4 + 32656Rr^5 - 44512r^6 \end{pmatrix}(s^2 - 4R^2 - 4Rr - 3r^2)^2$$
$$\Leftrightarrow 5015115R^{10} + 3286512R^9r - 4148388R^8r^2 - 2810656R^7r^3 + 3158640R^6r^4$$
$$+4576384R^5r^5 + 1864304R^4r^6 - 30576R^3r^7 + 461947R^2r^8$$
$$+515616Rr^9 + 266688r^{10} \overset{(****)}{\geq}$$
$$\begin{pmatrix} 1171260R^8 - 205632R^7r - 1379376R^6r^2 + 499296R^5r^3 + \\ 845204R^4r^4 - 101852R^3r^5 - 57255R^2r^6 + 79568Rr^7 + 133024r^8 \end{pmatrix}s^2$$

$$\text{Now, } 1171260R^8 - 205632R^7r - 1379376R^6r^2 + 499296R^5r^3 + 845204R^4r^4$$
$$-101852R^3r^5 - 57255R^2r^6 + 79568Rr^7 + 133024r^8$$
$$= (R - 2r)\begin{pmatrix} 1171260R^7 + 2136888R^6r + 2894400R^5r^2 + 6288096R^4r^3 + \\ 13421396R^3r^4 + 26740940R^2r^5 + 53424625Rr^6 + 106928818r^7 \end{pmatrix}$$
$$+213990660r^8 \overset{\text{Euler}}{\geq} 213990660r^8 > 0 \therefore \text{RHS of } (****) \overset{\text{Blundon-Gerretsen}}{\leq}$$
$$\begin{pmatrix} 1171260R^8 - 205632R^7r - 1379376R^6r^2 + 499296R^5r^3 + 845204R^4r^4 \\ -101852R^3r^5 - 57255R^2r^6 + 79568Rr^7 + 133024r^8 \end{pmatrix} \cdot \frac{R(4R + r)^2}{4R - 2r}$$
$$\overset{?}{\leq} \text{LHS of } (****)$$
$$\Leftrightarrow 1320300t^{11} - 2964150t^{10} - 622764t^9 + 306056t^8 + 2117616t^7$$
$$+6356960t^6 - 809860t^5 - 4564108t^4 - 798733t^3 - 5190t^2$$
$$-97504t - 533376 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t - 2)\left((t - 2)\begin{pmatrix} 1320300t^9 + 2317050t^8 + 3364236t^7 + 4494800t^6 + \\ 6639872t^5 + 14937248t^4 + 312379644t^3 + 65205476t^2 \\ +130504595t + 261191286 \end{pmatrix} + 522649260\right)$$
$$\overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{a}{9R^2 - a^2} + \frac{b}{9R^2 - b^2} + \frac{c}{9R^2 - c^2} \leq \frac{9}{2\sqrt{3(a^2 + b^2 + c^2)}}$$
$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$