ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, p_b, p_c $\to$ Spieker's cevians, the following relationship holds : $p_a + p_b + p_c \le m_a + m_b + m_c + \frac{1}{3}(max\{a, b, c\} - min\{a, b, c\})$

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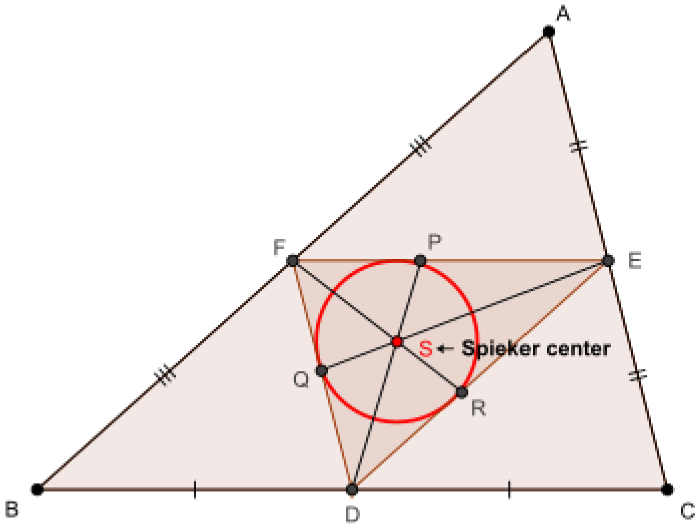

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at : $AS^2 =$

$$\frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} = \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}}$$

$$+ \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Again, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$
$$\text{Also, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$
$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$
$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta$$
$$=rs\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$
$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$
$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$
$$=\frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$
$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

Now, we shall prove that :

$$\frac{1}{2}(|b-c|+|c-a|+|a-b|) \overset{(\blacklozenge)}{=} max\{a,b,c\} - \min\{a,b,c\}$$

$\boxed{\text{Case (1)}}$ $a \ge b \ge c \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(b-c+a-c+a-b)$

$= a - c = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (2)}}$ $a \ge c \ge b \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(c-b+a-c+a-b)$

$= a - b = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (3)}}$ $b \ge c \ge a \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(b-c+c-a+b-a)$

$= b - a = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (4)}}$ $b \ge a \ge c \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(b-c+a-c+b-a)$

$= b - c = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (5)}}$ $c \ge a \ge b \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(c-b+c-a+a-b)$

$= c - b = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (6)}}$ $c \ge b \ge a \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(c-b+c-a+b-a)$

$= c - a = max\{a,b,c\} - \min\{a,b,c\}$

$\therefore$ combining all 6 cases, we conclude : $\frac{1}{2}(|b-c|+|c-a|+|a-b|)$

$= max\{a,b,c\} - \min\{a,b,c\}$

$$\therefore \text{via } (\blacklozenge), p_a + p_b + p_c \le m_a + m_b + m_c + \frac{1}{3}(max\{a,b,c\} - \min\{a,b,c\})$$

$$\Leftrightarrow \sum_{cyc} p_a \overset{(\blacksquare)}{\le} \sum_{cyc} m_a + \frac{1}{6}\sum_{cyc}|b-c|$$

$$\text{Now, } a^2(4m_a^2) = a^2(2b^2+2c^2-a^2)$$

$$= 2a^2b^2 + 2c^2a^2 + 2b^2c^2 - a^4 - b^4 - c^4 + (b^4 + c^4 - 2b^2c^2)$$

$$= 16F^2 + (b^2-c^2)^2 > (b^2-c^2)^2 \Rightarrow 2am_a > |b^2-c^2|$$

$$\Rightarrow m_a > \frac{|b-c|(b+c)}{2a} \therefore m_a > \frac{|b-c|(2s-a)}{2a} \Rightarrow \left(m_a + \frac{|b-c|}{6}\right)^2$$

$$= m_a^2 + \frac{(b-c)^2}{36} + m_a.\frac{|b-c|}{3} \ge m_a^2 + \frac{(b-c)^2}{36} + \frac{|b-c|(2s-a)}{2a}.\frac{|b-c|}{3} \overset{?}{\ge} p_a^2$$

$$\Leftrightarrow \frac{(2s-a)(b-c)^2}{6a} \overset{?}{\ge} p_a^2 - m_a^2 - \frac{(b-c)^2}{36} \overset{\text{via }(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} - \frac{(b-c)^2}{36}$$

$$= \frac{9(8s^2-a^2)-(2s+a)^2}{36(2s+a)^2}.(b-c)^2 = \frac{68s^2-4sa-10a^2}{36(2s+a)^2}.(b-c)^2$$

$$\Leftrightarrow 3(2s-a)(2s+a)^2 \overset{?}{\ge} a(34s^2-2sa-5a^2) \left(\because (b-c)^2 \ge 0\right)$$

$$\Leftrightarrow 12s^3 - 11s^2a - 2sa^2 + a^3 \overset{?}{\ge} 0 \Leftrightarrow (s-a)\left(12s^2 + a(s-a)\right) \overset{?}{\ge} 0 \rightarrow \text{true}$$

$$(\text{strict}) \because s > a \therefore m_a + \frac{|b-c|}{6} \geq p_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} m_a + \frac{1}{6}\sum_{cyc}|b-c| \geq \sum_{cyc} p_a \Rightarrow (\blacksquare) \text{ is true}$$

$$\therefore p_a + p_b + p_c \leq m_a + m_b + m_c + \frac{1}{3}(\max\{a,b,c\} - \min\{a,b,c\})$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$