ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, p_b, p_c

$\to$ Spieker cevians, the following relationship holds :

$$h_a(p_a+w_a)+h_b(p_b+w_b)+h_c(p_c+w_c)\le 2s^2$$

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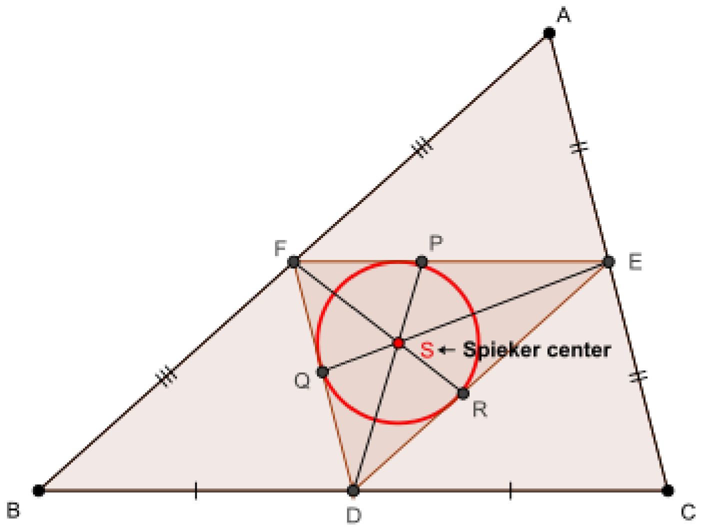

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$
$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$
$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES},\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$
$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$
$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Also, } p_a^2-m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2 \overset{(\blacksquare\blacksquare)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a$$

$$\therefore \text{in order to prove : } \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a}, \text{it suffices to prove :}$$

$$p_a^2-m_a^2 \leq m_a^2-w_a^2$$

$$\overset{\text{via }(\blacksquare\blacksquare)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \leq s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$= \frac{(b-c)^2}{4}\left(1+\frac{4s(s-a)}{(2s-a)^2}\right) = \frac{(b-c)^2}{4}\cdot\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow \left((2s-a)^2+4s(s-a)\right)(2s+a)^2 \geq (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3-12s^2a-4sa^2+a^3 \geq 0 \Leftrightarrow (s-a)(16s^2+4sa)+a^3 \geq 0$$

$$\rightarrow \text{true (strict) since } s>a \therefore \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a} \Rightarrow p_a+w_a \leq 2m_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} h_a(p_a+w_a) \leq 4rs\sum_{cyc}\frac{m_a}{a} \overset{?}{\leq} 2s^2 \Leftrightarrow \frac{s}{2r} \overset{?}{\geq} \sum_{cyc}\frac{m_a}{a} \Leftrightarrow \frac{s}{2r} \overset{?}{\geq} \frac{1}{4Rrs}\sum_{cyc} bcm_a$$

$$\Leftrightarrow (2Rs^2)^2 \overset{?}{\geq} \left(\sum_{cyc} bcm_a\right)^2$$

$$= \sum_{cyc}\left(b^2c^2\left(\frac{2b^2+2c^2-a^2}{4}\right)\right)+2\sum_{cyc}(bc.ca.m_am_b)$$

$$\Leftrightarrow 16R^2s^4 \overset{?}{\geq} \sum_{cyc}\left(b^2c^2\left(2\sum_{cyc}a^2-3a^2\right)\right)+32Rrs\sum_{cyc} cm_am_b$$

$$\Leftrightarrow 16R^2s^4 \underset{(\bullet)}{\overset{?}{\geq}} 4(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)$$

$$-144R^2r^2s^2+32Rrs\sum_{cyc} cm_am_b$$

$$\text{Now, } m_am_b \overset{?}{\leq} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right) \overset{?}{\leq} \frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \overset{?}{\geq} 0 \to \text{true} \Rightarrow m_am_b \leq \frac{2c^2+ab}{4} \text{ and analogs}$$

$$\therefore \text{RHS of } (\bullet) \leq 4(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-144R^2r^2s^2$$

$$+32Rrs\sum_{cyc}\left(c.\frac{2c^2+ab}{4}\right) \overset{?}{\leq} 16R^2s^4$$

$$\Leftrightarrow (s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-36R^2r^2s^2$$

$$+8Rrs(4(s^2-6Rr-3r^2)+12Rrs) \overset{?}{\leq} 4R^2s^4$$

$$\Leftrightarrow s^6-(4R^2+4Rr-r^2)s^4-(12R^2+16Rr+r^2)r^2s^2-r^3(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\leq}} 0$$

Now, Rouche $\Rightarrow s^2-(m-n) \geq 0$ and $s^2-(m+n) \leq 0$, where $m =$
$2R^2+10Rr-r^2$ and $n = 2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2 \leq 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \leq 0$$

$\Rightarrow s^6-(4R^2+20Rr-2r^2)s^4+r(4R+r)^3.s^2 \leq 0 \therefore$ in order to prove $(\bullet\bullet)$,
it suffices to prove :

$$s^6-(4R^2+4Rr-r^2)s^4-(12R^2+16Rr+r^2)r^2s^2-r^3(4R+r)^3$$
$$\leq s^6-(4R^2+20Rr-2r^2)s^4+r(4R+r)^3.s^2$$

$$\Leftrightarrow (16R-5r)s^4-(64R^3+60R^2r+28Rr^2+2r^3)s^2-r^2(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\leq} 0$$

$$\text{Again, LHS of } (\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\leq}$$

$$\left((16R-5r)(4R^2+4Rr+3r^2)-(64R^3+60R^2r+28Rr^2+2r^3)\right)s^2$$

$$-r^2(4R+r)^3 \overset{?}{\leq} 0 \Leftrightarrow (16R-5r)s^2 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\leq}} (4R+r)^3$$

$$\text{Moreover, LHS of } (\bullet\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\leq} (16R-5r)(4R^2+4Rr+3r^2) \overset{?}{\leq} (4R+r)^3$$

$$\Leftrightarrow 4r(R-2r)^2 \overset{?}{\geq} 0 \to \text{true} \Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore h_a(p_a+w_a)+h_b(p_b+w_b)+h_c(p_c+w_c) \leq 2s^2$$

$\forall \Delta ABC$, " $=$ " iff ΔABC is equilateral (QED)