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In any Δ ABC with p_a, p_b, p_c
$\to$ Spieker cevians, the following relationship holds :

$$\frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c}\geq 3$$

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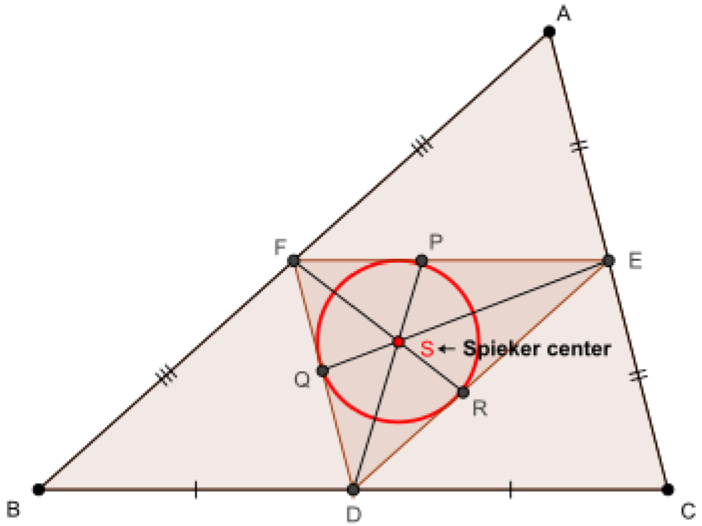

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{We have}: \prod_{cyc}(2s+a) = 8s^3 + 4s^2\sum_{cyc} a + 2s\sum_{cyc} ab + 4Rrs$$
$$= 8s^3 + 4s^2.2s + 2s(s^2+4Rr+r^2) + 4Rrs$$
$$\Rightarrow \prod_{cyc}(2s+a) \overset{(\blacksquare\blacksquare)}{=} 2s(9s^2+6Rr+r^2)$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$
$$= \sum_{cyc} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$
$$\Rightarrow b^3+c^3-abc+a(4m_a^2) \overset{(\blacksquare\blacksquare\blacksquare)}{=} 2s(s^2-8Rr-3r^2+8Rr\cos A) \therefore (\blacksquare),(\blacksquare\blacksquare\blacksquare)$$
$$\Rightarrow p_a = \frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}$$
$$= \frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\left(1-2\sin^2\frac{A}{2}\right)}$$
$$\Rightarrow p_a = \frac{2s}{2s+a}.\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}} \text{ and analogs}$$
$$\Rightarrow \frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c} = \sum_{cyc}\frac{(2s+a)r_a}{2s.\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}}$$
$$= \frac{\prod_{cyc}(2s+a)}{2s}.\sum_{cyc}\frac{r_a}{\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}(2s+b)(2s+c)}$$
$$\overset{\text{via }(\blacksquare\blacksquare)}{=} \frac{2s(9s^2+6Rr+r^2)}{2s}.$$
$$\sum_{cyc}\frac{r_a^2}{\sqrt{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(2s+b)(2s+c)r_a}.\sqrt{(2s+b)(2s+c)r_a}}$$
$$\overset{\text{Bergstrom}}{\geq}$$
$$\left(\begin{matrix}9s^2+6Rr\\+r^2\end{matrix}\right).\frac{\left(\sum_{cyc} r_a\right)^2}{\sum_{cyc}\left(\sqrt{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(2s+b)(2s+c)r_a}.\sqrt{(2s+b)(2s+c)r_a}\right)}$$
$$\overset{\text{CBS}}{\geq} \frac{(9s^2+6Rr+r^2)(4R+r)^2}{\sqrt{\sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(2s+b)(2s+c)r_a\right)}.\sqrt{\sum_{cyc}((2s+b)(2s+c)r_a)}}$$
$$\Rightarrow \frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c} \overset{(\blacklozenge)}{\geq}$$
$$\frac{(9s^2+6Rr+r^2)(4R+r)^2}{\sqrt{\sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(8s^2-2sa+bc)r_a\right)}.\sqrt{\sum_{cyc}((8s^2-2sa+bc)r_a)}}$$

$$\text{We have}: \sum_{cyc}\frac{r_a}{a} = \frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}} = \frac{s}{4R}\sum_{cyc}\sec^2\frac{A}{2} = \frac{s}{4R}.\frac{s^2+(4R+r)^2}{s^2}$$

$$\Rightarrow \sum_{cyc} \frac{r_a}{a} \overset{([redacted])}{=} \frac{s^2+(4R+r)^2}{4Rs} \text{ and } \sum_{cyc} ar_a = rs\sum_{cyc}\frac{a-s+s}{s-a}$$

$$= rs\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right) \Rightarrow \sum_{cyc} ar_a \overset{([redacted])}{=} (4R-2r)s$$

We now proceed to evaluate : $\sum_{cyc}\left(-16Rr\sin^2\frac{A}{2}(8s^2-2sa+bc)r_a\right)$

Firstly, $\sum_{cyc}\left(-8s^2.16Rr\sin^2\frac{A}{2}r_a\right) = -128Rrs^2\sum_{cyc}\left(\left(1-\cos^2\frac{A}{2}\right)s\tan\frac{A}{2}\right)$

$$= -128Rrs^2(4R+r) + 32rs^3\sum_{cyc}\left(4R\cos\frac{A}{2}\sin\frac{A}{2}\right)$$

$$= -128Rrs^2(4R+r) + 32rs^3(2s)$$

$$\therefore \sum_{cyc}\left(-8s^2.16Rr\sin^2\frac{A}{2}r_a\right) = -128Rrs^2(4R+r) + 64rs^4 \rightarrow (3)$$

Secondly, $\sum_{cyc}\left(\left(-16Rr\sin^2\frac{A}{2}\right)(-2sa)\right) = 32Rrs\sum_{cyc}\left(\left(1-\cos^2\frac{A}{2}\right)ar_a\right)$

$$= 32Rrs\left(\sum_{cyc} ar_a - \sum_{cyc}\left(a.s\tan\frac{A}{2}.\cos^2\frac{A}{2}\right)\right) \overset{\text{via } ([redacted])}{=}$$

$$32Rrs\left((4R-2r)s - \frac{sa}{4R}.\left(4R\cos\frac{A}{2}\sin\frac{A}{2}\right)\right)$$

$$= 32Rrs^2(4R-2r) - 8rs^2\sum_{cyc} a^2 = 32Rrs^2(4R-2r) - 16rs^2(s^2-4Rr-r^2)$$

$$\therefore \sum_{cyc}\left(\left(-16Rr\sin^2\frac{A}{2}\right)(-2sa)\right) = 32Rrs^2(4R-2r) - 16rs^2(s^2-4Rr-r^2) \rightarrow (4)$$

Thirdly, $\sum_{cyc}\left(-16Rr\sin^2\frac{A}{2}bcr_a\right) = -16Rr.4Rrs.\sum_{cyc}\frac{\sin^2\frac{A}{2}.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}.\tan\frac{A}{2}}$

$$= -16Rr^2\sum_{cyc} r_a^2 \therefore \sum_{cyc}\left(-16Rr\sin^2\frac{A}{2}bcr_a\right) = -16Rr^2\left((4R+r)^2-2s^2\right) \rightarrow (5)$$

Again, $\sum_{cyc}\left((8s^2-2sa+bc)r_a\right) = 8s^2(4R+r) - 2s\sum_{cyc} ar_a + 4Rrs.\sum_{cyc}\frac{r_a}{a}$

$$\overset{\text{via } ([redacted]) \text{ and } ([redacted])}{=} 8s^2(4R+r) - 2s^2(4R-2r) + 4Rrs.\frac{s^2+(4R+r)^2}{4Rs}$$

$$\therefore \sum_{cyc}\left((8s^2-2sa+bc)r_a\right) = (24R+13r)s^2 + r(4R+r)^2 \rightarrow (6)$$

Via (3), (4), (5), (6), $\sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(8s^2-2sa+bc)r_a\right)$

$$= (s^2-3r^2)\left((24R+13r)s^2+r(4R+r)^2\right) - 128Rrs^2(4R+r) + 64rs^4$$
$$+32Rrs^2(4R-2r) - 16rs^2(s^2-4Rr-r^2) - 16Rr^2\left((4R+r)^2-2s^2\right)$$

$$\therefore \sum_{cyc}\left(\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)(8s^2-2sa+bc)r_a\right)$$

$$= (24R+61r)s^4 - r(368R^2+160Rr+22r^2)s^2$$
$$-r^2(256R^3+176R^2r+40Rr^2+3r^3) \to (7)$$
$$\therefore (6),(7)\ and\ (\blacklozenge) \Rightarrow \frac{r_a}{p_a}+\frac{r_b}{p_b}+\frac{r_c}{p_c} \geq$$
$$\frac{(9s^2+6Rr+r^2)(4R+r)^2}{\sqrt{(24R+61r)s^4 - r(368R^2+160Rr+22r^2)s^2 - r^2(256R^3+176R^2r+40Rr^2+3r^3)}}.$$
$$\frac{1}{\sqrt{(24R+13r)s^2+r(4R+r)^2}} \overset{?}{\geq} 3$$
$$\Leftrightarrow (9s^2+6Rr+r^2)^2(4R+r)^4 \overset{?}{\geq}$$
$$9\begin{pmatrix}(24R+61r)s^4 - r(368R^2+160Rr+22r^2)s^2\\ -r^2(256R^3+176R^2r+40Rr^2+3r^3)\end{pmatrix}\left((24R+13r)s^2+r(4R+r)^2\right)$$
$$\Leftrightarrow -(5184R^2+15984Rr+7137r^2)s^6$$
$$+(20736R^4+96768R^3r+74880R^2r^2+20160Rr^3+2106r^4)s^4$$
$$+r(27648R^5+140544R^4r+132480R^3r^2+50688R^2r^3+8748Rr^4+567r^5)s^2$$
$$+r^2\begin{pmatrix}9216R^6+49152R^5r+50560R^4r^2+22720R^3r^3+5220R^2r^4+604Rr^5\\ +28r^6\end{pmatrix} \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 0$$

Now, Rouche $\Rightarrow s^2-(m-n) \geq 0$ and $s^2-(m+n) \leq 0$, where $m =$ $2R^2+10Rr-r^2$ and $n = 2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \leq 0$$
$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3 \leq 0$$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \geq$

$$-(5184R^2+15984Rr+7137r^2)(s^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$
$$-4r(17712R^3+65745R^2r+22653Rr^2-4095r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\ +r(4R+r)^3\end{pmatrix}$$
$$\Leftrightarrow (9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4-3132r^5)s^2$$
$$+r\begin{pmatrix}567936R^6+2535072R^5r+2415368R^4r^2+818798R^3r^3+71163R^2r^4\\ -13168Rr^5-2044r^6\end{pmatrix} \boxed{\overset{(\bullet\bullet)}{\geq}} 0$$

$\boxed{Case\ 1}$ $9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4$ $-3132r^5 \geq 0$ *and then* : LHS of $(\bullet\bullet) \geq$

$$\begin{pmatrix}9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4\\ -3132r^5\end{pmatrix}(16Rr-5r^2)$$
$$+r\begin{pmatrix}567936R^6+2535072R^5r+2415368R^4r^2+818798R^3r^3+71163R^2r^4\\ -13168Rr^5-2044r^6\end{pmatrix} \overset{?}{\geq} 0$$
$$\Leftrightarrow 360000t^6+187248t^5-2523815t^4+1048165t^3+851814t^2-225140t$$
$$+6808 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}360000t^4+1627248t^3+2545177t^2\\ +4719881t+9550630\end{pmatrix}+19097856\right) \overset{?}{\geq} 0$$

$\to$ true $\because t \overset{Euler}{\geq} 2 \Rightarrow (\bullet\bullet)$ is true

$\boxed{Case\ 2}$ $9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3+77400Rr^4$ $-3132r^5 < 0$ *and then* : LHS of $(\bullet\bullet) \geq$

$$\begin{pmatrix}9504R^5-132066R^4r-507708R^3r^2-78813R^2r^3\\ +77400Rr^4-3132r^5\end{pmatrix}(4R^2+4Rr+3r^2)$$
$$+r\begin{pmatrix}567936R^6+2535072R^5r+2415368R^4r^2+818798R^3r^3+71163R^2r^4\\ -13168Rr^5-2044r^6\end{pmatrix} \overset{?}{\geq} 0$$

$$\Leftrightarrow 19008t^7 + 38844t^6 + 2244t^5 - 163457t^4 - 354989t^3$$

$$+65898t^2 + 103252t - 5720 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)\left((t-2)\begin{pmatrix}19008t^5 + 114876t^4 + 385716t^3 + 919903t^2\\ +1781759t + 3513322\end{pmatrix} + 7029504\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \text{ is true} \therefore \text{combining both cases}, (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\forall \Delta ABC \therefore \frac{r_a}{p_a} + \frac{r_b}{p_b} + \frac{r_c}{p_c} \geq 3 \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$