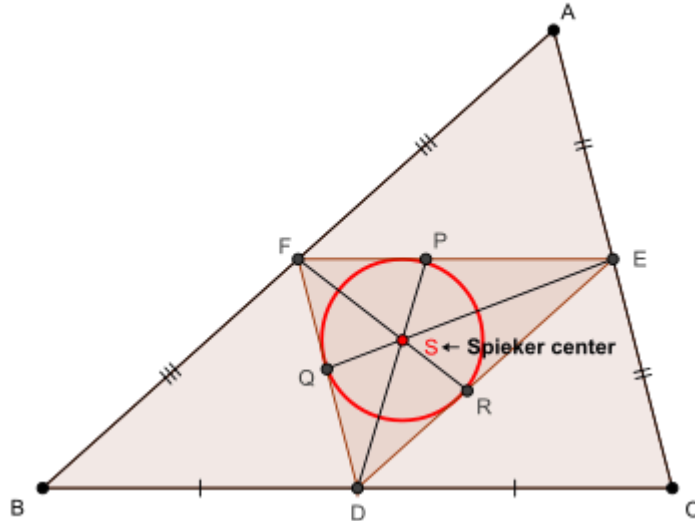


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In any acute $\triangle ABC$ with p_a, p_b, p_c
 \rightarrow Spieker cevians, the following relationship
 holds : $r_a(p_a + w_a) + r_b(p_b + w_b) + r_c(p_c + w_c) \leq 2s^2$

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Let AS produced meet BC at X and $m(\sphericalangle BAX) = \alpha$ and $m(\sphericalangle CAX) = \beta$ (say)
 and inradius of $\triangle DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \triangle DEF, \therefore m(\sphericalangle AFS) = B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2}$$

$$= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\sphericalangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on $\triangle AFS$ and $\triangle AES$, we arrive at :

$$AS^2 = \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2}$$

$$= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} + \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2}$$

$$\begin{aligned}
 & \text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 &= \frac{r}{2} \left(4R\cos\frac{C}{2} \sin\frac{A-B}{2} + 4R\cos\frac{B}{2} \sin\frac{A-C}{2} \right) \\
 &= Rr \left(2\sin\frac{A+B}{2} \sin\frac{A-B}{2} + 2\sin\frac{A+C}{2} \sin\frac{A-C}{2} \right) \\
 &= Rr \left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left(1 - 2\sin^2\frac{A}{2} \right) \right) \\
 &= 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 &= \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 &= \frac{4(b+c)bcsin^2\frac{A}{2} - 2a \cdot 2bccosA}{8s} = \frac{bc \left((2s-a)\sin^2\frac{A}{2} - a \left(1 - 2\sin^2\frac{A}{2} \right) \right)}{2s} \\
 &= \frac{bc \left((2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 &\Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 &\quad \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 & \text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 &= \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} \\
 & \text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 &= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s} \\
 &= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 & \text{Via sine law on } \triangle AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{4s}{(a+b)\sin\frac{C}{2}} \\
 &\Rightarrow c\sin\alpha \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS} \\
 & \text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs \\
 & \text{via (***) and (***)} \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS \\
 &\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s} \\
 &\therefore p_a^2 \stackrel{(\blacksquare)}{=} \frac{2s}{(2s+a)^2} (b^3+c^3-abc+a(4m_a^2))
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } p_a^2 - m_a^2 &= \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2)) - m_a^2 \\
 &= \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc) - \left(1 - \frac{8sa}{(2s+a)^2}\right) m_a^2 \\
 &= \frac{4(a+b+c)(b^3 + c^3 - abc) - (2b^2 + 2c^2 - a^2)(b+c)^2}{4(2s+a)^2} \\
 &= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2 - c^2)^2}{4(2s+a)^2} \\
 &= \frac{(b-c)^2}{4(2s+a)^2} \left((a^2 + 2a(b+c) + (b+c)^2) + ((b+c)^2 + 2a(b+c) + a^2) - a^2 \right) \\
 &= \frac{(b-c)^2}{4(2s+a)^2} (2(a+b+c)^2 - a^2) = \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2} \\
 \therefore p_a^2 - m_a^2 &\stackrel{(\blacksquare)}{=} \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2} \stackrel{s > a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a \\
 \therefore \text{in order to prove: } &\frac{p_a^2 - m_a^2}{p_a + m_a} \leq \frac{m_a^2 - w_a^2}{m_a + w_a}, \text{ it suffices to prove:} \\
 &\frac{p_a^2 - m_a^2}{p_a + m_a} \leq m_a^2 - w_a^2 \\
 \text{via } (\blacksquare) \Leftrightarrow &\frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2} \leq s(s-a) + \frac{(b-c)^2}{4} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2} \right) \\
 &= \frac{(b-c)^2}{4} \left(1 + \frac{4s(s-a)}{(2s-a)^2} \right) = \frac{(b-c)^2}{4} \cdot \frac{(2s-a)^2 + 4s(s-a)}{(2s-a)^2} \\
 &\Leftrightarrow ((2s-a)^2 + 4s(s-a))(2s+a)^2 \geq (8s^2 - a^2)(2s-a)^2 \\
 &\Leftrightarrow 16s^3 - 12s^2a - 4sa^2 + a^3 \geq 0 \Leftrightarrow (s-a)(16s^2 + 4sa) + a^3 \geq 0 \\
 \rightarrow \text{true (strict) since } s > a &\therefore \frac{p_a^2 - m_a^2}{p_a + m_a} \leq \frac{m_a^2 - w_a^2}{m_a + w_a} \Rightarrow p_a + w_a \leq 2m_a \text{ and analogs} \\
 &\Rightarrow \sum_{\text{cyc}} r_a(p_a + w_a) \leq 2 \sum_{\text{cyc}} m_a r_a \leq \\
 2 \sum_{\text{cyc}} R(1 + \cos A)r_a & (\because m_a \leq R(1 + \cos A) \text{ and analogs } \forall \text{ acute triangles}) \\
 &= 2R \sum_{\text{cyc}} \frac{2s(s-a)}{bc} \cdot \frac{rs}{s-a} = \frac{4Rrs^2}{4Rrs} \cdot \sum_{\text{cyc}} a = 2s^2 \\
 \therefore r_a(p_a + w_a) + r_b(p_b + w_b) + r_c(p_c + w_c) &\leq 2s^2 \\
 \forall \text{ acute } \triangle ABC, '' = '' \text{ iff } \triangle ABC \text{ is equilateral (QED)} &
 \end{aligned}$$